# A Flux-form version of the Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLAM)

**Peter H. Lauritzen** [a,*] **and Lucas M. Harris**[b]

[a] **pel@ucar.edu**, *Climate and Global Dynamics Division, National Center for Atmospheric Research (NCAR), Boulder, Colorado, USA.*

[b] **lharris@atmos.washington.edu**, *Department of Atmospheric Sciences, University of Washington, Seattle, Washington, USA.*

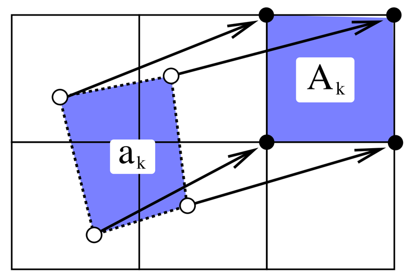
* *Supported by* **DOE BER program** *#DE-SC0001658*

## Lagrangian formulation of CSLAM

The 2D transport equation (no sources/sinks) for cell $A_k$ is given by:

$$\frac{d}{dt}\int_{A_k(t)} \psi \, dA = 0, \qquad \textbf{Lagrangian} \tag{1}$$

where $\psi$ is the density, $dA$ is the element area, and $A_k(t)$ Lagrangian area.




Use upstream discretization:

- $A_k(t+\Delta t) = A_k$ = Eulerian grid cell with area $\Delta A_k$
- $A_k(t) = a_k$ = corresponding upstream Lagrangian (deformed) cell with area $\delta a_k$

Then (1) can be written as

$$\overline{\psi}_k^{n+1} \Delta A_k = \overline{\psi_k^*}^n \, \delta a_k \tag{2}$$

where $\overline{(\cdot)}^{n+1}$ is the average of $\psi$ over $A_k$ and

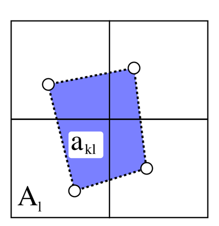
$$\overline{\psi_k^*}^n = \frac{1}{\delta a_k} \iint_{a_k} \psi^n(x,y)\, dx\, dy. \tag{3}$$

- Use polynomial reconstruction functions in each Eulerian cell $\ell$ in the form

$$f_\ell(x,y) = \sum_{i+j\le 2} c_\ell^{(i,j)} x^i y^j, \quad i,j \in \{0,1,2\}, \tag{4}$$

  where $c_\ell^{(i,j)}$ are derived coefficients ensuring mass-conservation.
- Reconstructions are piecewise parabolic, and so **third order accurate**.
- Since $f_\ell(x,y)$ is local to $A_\ell$ and no continuity across cell borders is enforced, the integration over $a_k$ in (5) must be split into overlap integrals

$$\overline{\psi_k^*}^n = \frac{1}{\delta a_k} \sum_{\ell=1}^{L_k} \iint_{a_{k\ell}} f_\ell(x,y)\, dx\, dy. \tag{5}$$




where $L_k$ is the number of overlap areas and

$$a_{k\ell} = a_k \cap A_\ell, \quad a_{k\ell} \neq \emptyset; \quad \ell = 1,..,L_k.$$

- We convert area integrals in (5) into line-integrals by applying the Gauss-Green theorem (Dukowicz 1984):

$$\iint_{a_{k\ell}} f_\ell(x,y)\, dx\, dy = \oint_{\partial a_{k\ell}} [P\, dx + Q\, dy],$$

  where $\partial a_{k\ell}$ is the boundary of $a_{k\ell}$ and $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = f_\ell(x,y)$.
- $\Rightarrow$ CSLAM scheme is given by (Lauritzen et al. 2010)

$$\overline{\psi}_k^{n+1} \Delta A_k = \sum_{\ell=1}^{L_k} \iint_{a_{k\ell}} f_\ell(x,y)\, dx\, dy = \sum_{\ell=1}^{L_k} \left[ \sum_{i+j\le 2} c_\ell^{(i,j)} w_{k\ell}^{(i,j)} \right], \tag{6}$$

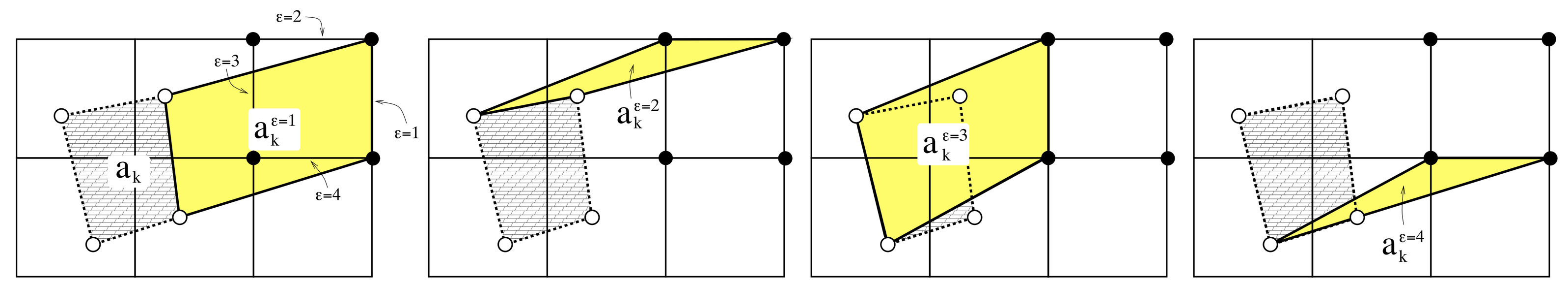
  where $w_{k\ell}^{(i,j)}$ are functions of the coordinate locations of the vertices of $a_{k\ell}$ and can be re-used for each additional tracer (**multi-tracer efficiency**).

## Flux-form formulation of CSLAM (FF-CSLAM)

The two-dimensional transport equation (no sources/sinks) for a cell $A_k$:

$$\frac{d}{dt}\int_{A_k} \psi \, dA + \oint_{\partial A_k} (\psi \vec{v}) \cdot \vec{n}\, dS = 0, \qquad \textbf{flux-form (Eulerian)} \tag{7}$$

where $\partial A_k$ is the boundary of static Eulerian cell $A_k$ and $\vec{n}$ the outward normal vector to $\partial A_k$. The second-term on the left-hand side of (7) represents the instantaneous flux of mass through the boundaries of $A_k$.




Flux-form version of CSLAM is based on semi-Lagrangian discretization of (7)

$$\overline{\psi}_k^{n+1} \Delta A_k = \overline{\psi}_k^{n} \Delta A_k + \sum_{\varepsilon=1}^{4} s^\varepsilon F_k^\varepsilon, \tag{8}$$

where $s^\varepsilon = \pm 1$ for inflow and outflow, respectively, and face-fluxes

$$F_k^\varepsilon = \sum_{\ell=1}^{L_k^\varepsilon} F_{k\ell}^\varepsilon = \sum_{\ell=1}^{L_k^\varepsilon} \iint_{a_k^\varepsilon} f_\ell(x,y)\, dx\, dy, \tag{9}$$

where $L_k^\varepsilon$ is the number of non-empty overlap areas between the flux-area $a_k^\varepsilon$ and the Eulerian grid. More details in Harris and Lauritzen (2010).

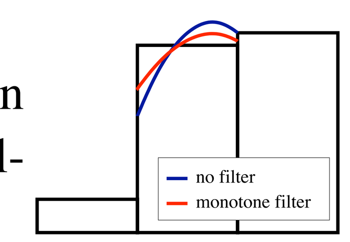
Note that the union of the areas used in FF-CSLAM and CSLAM are identical

$$\Delta A - \sum_{\varepsilon=1}^{4} (s^\varepsilon \times \delta a_k^\varepsilon) = \delta a_k,$$

i.e. the two schemes will produce identical results.

- Extension to the cubed-sphere is described in Lauritzen et al. (2010).
- Method extendable to other unstructured spherical grids (e.g. icosahedral).

## Monotonicity preservation

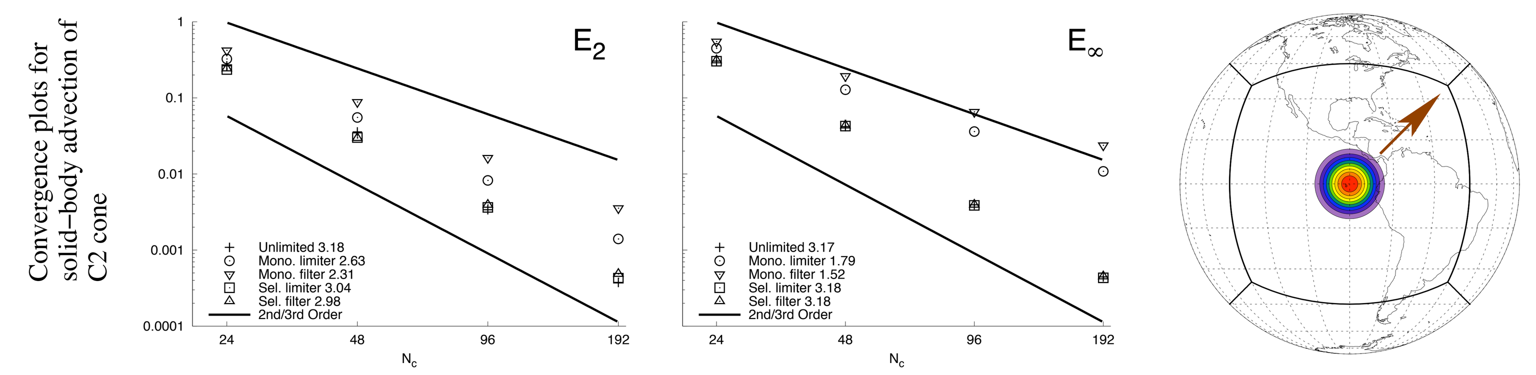



- A priori ("Monotone filtering"): Filter the reconstruction $f_\ell(x,y)$ so that extreme values lie within the adjacent cell-average values.
- A posteriori ("Monotone limiting"): Limit the fluxes to prevent new extrema in $\overline{\psi}^{n+1}$ using flux-corrected transport (Zalesak 1979).
- Selective filtering/selective limiting (Blossey and Durran 2008): apply filtering or limiting only where a smoothness metric exceeds a certain threshold.

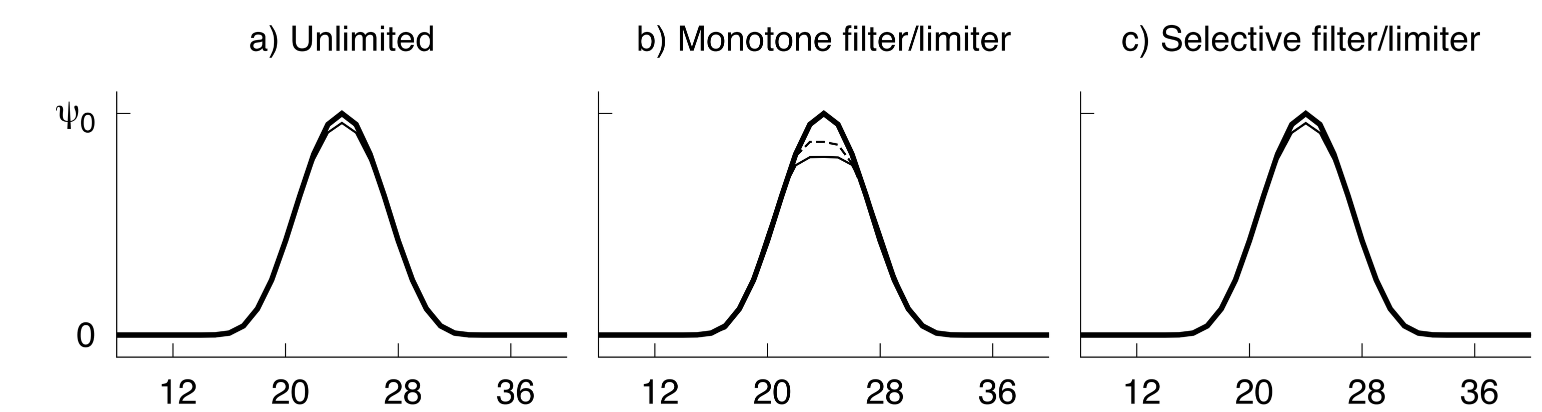
## Why flux-form?

- FF-CSLAM can use flux limiters and flux-corrected transport (Zalesak 1979) as well as reconstruction filtering to enforce shape-preservation (monotonicity and positivity), while Lagrangian CSLAM is restricted to filtering.
- Flux-form methods allow **sub-cycling**, or tracer advection at a longer timestep than that of the dynamics. This is done by accumulating the air-mass fluxes for the continuity equation over the long timestep and multiplying by the time- and space-averaged mixing ratio for the tracer.

## Results: smooth cosine bell test




- A smooth cosine bell tracer distribution was advected once around the sphere, over the corners of the cubed-sphere grid.
- The selective methods converge with third-order accuracy (see above) and retain physical extrema (see below), unlike either monotone filtering or monotone limiting.




**Efficiency** (for one tracer)

- For CFL<1 FF-CSLAM is at most 40% more expensive than CSLAM (for CFL≈3 FF-CSLAM is approximately 110% more expensive than CSLAM)
- The monotone filtering is much less efficient than monotone limiting in FF-CSLAM (monotone limiting almost doubles the cost whereas monotone filtering almost triples the cost)
- Selective filtering is more efficient than monotone limiting!

DOE BER effort: **Implement (FF-)CSLAM into CAM-HOMME**

### References

Blossey, P. and D. Durran, 2008: Selective monotonicity preservation in scalar advection. *J. Comput. Phys.*, **227**, 5160–5183.

Dukowicz, J., 1984: Conservative rezoning (remapping) for general quadrilateral meshes. *J. Comput. Phys.*, **54**, 411–424.

Harris, L. M. and P. H. Lauritzen, 2010: A flux-form version of the conservative semi-lagrangian multi-tracer transport scheme (cslam) on the cubed sphere grid. *J. Comput. Phys.*, submitted.

Lauritzen, P. H., R. D. Nair, and P. A. Ullrich, 2010: A conservative semi-Lagrangian multi-tracer transport scheme (cslam) on the cubed-sphere grid. *J. Comput. Phys.*, **229**, 1401–1424.

Zalesak, S. T., 1979: Fully multidimensional flux-corrected transport algorithms for fluids. *J. Comput. Phys.*, **31**, 335–362.