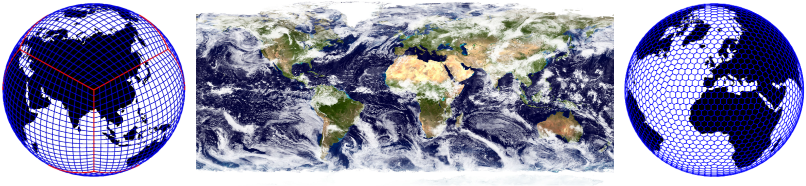
Some challenging idealized transport test cases for global models

Peter H. Lauritzen[1] Ramachandran D. Nair[2]

[1]Climate and Global Dynamics Division
National Center for Atmospheric Research (NCAR)

[2]Institute for Mathematics Applied to Geosciences (IMAGe), NCAR



Workshop on High-Resolution Global Modeling
Colorado State University, Fort Collins, Colorado

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1. Motivation
   - Why yet another idealized test case for transport?
2. Test case formulation:
   - Test case strategy
   - Class of test cases
   - Two non-divergent and two divergent tests
3. Some sample solutions
   - computed with new conservative semi-Lagrangian scheme
4. Advertisements
   - NCAR transport workshop for ‘transport geeks’ (early 2011)
   - Upcoming book: Numerical Techniques for Global Atmospheric Models

Why a new idealized test case?

Why idealized test cases in the first place?

- In "full-blown" model simulations: Difficult to relate features/deficiencies in the simulation to errors in the discretization schemes.
- $\Rightarrow$ Useful in model development to use idealized test cases where simplified settings make it easier to identify cause and effect

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What would we like to learn from idealized tests (linear transport)?

- Debug code (obvious bugs: non-conservation of symmetry, mass, ...)
- Accuracy:
  - absolute errors in terms of error norms (exact solution)
  - numerical convergence rates (do they match formal order of accuracy?)
  - efficiency = accuracy / computational cost (multi-tracer efficiency)
  - consistency: Monotonicity and preservation of relative concentrations (control chemical reaction rates!)
  - accuracy across scales (given the increased resolution span in global models): Flow should force grid-scale features from well-resolved initial conditions

Preservation of relative concentrations (chemistry)

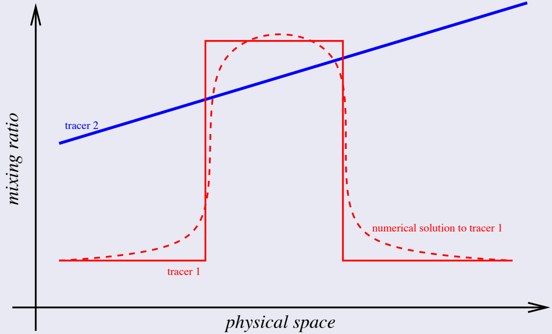
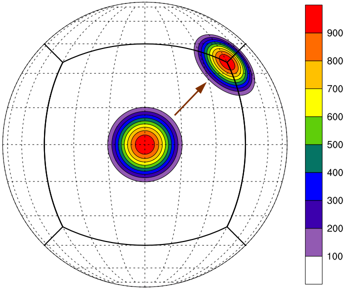


Figure:

- Assume constant wind
- Most schemes can transport tracer 2 exactly but not tracer 1
- $\rightarrow$ **Relative concentrations near large gradients are altered by numerical scheme ... can trigger highly non-linear irreversible processes**

Why a new idealized test case?

- **However**, according to the literature most transport schemes in global climate models have ‘only’ been tested (in idealized setup) for solid-body advection:
  - No flow features of much interest (no deformation, no divergence, etc.)
  - Parcel trajectories are along straight-circle arcs
  - Preservation of a constant is easier compared to more complex non-divergent flows (Lagrangian areas undergo no deformation, rotation, etc.)
  - Does not force modelers to distinguish between tracer concentration q and tracer density $\rho\, q$ (more discussion on next slide)



Why a new idealized test case?

- Air mass and tracer mass equations in flux-form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{1}$$

$$\frac{\partial (\rho\, q)}{\partial t} + \nabla \cdot (\rho\, q\, \vec{v}) = 0, \tag{2}$$

where $\vec{v}$ velocity, ρ density (kg/V), q concentration (kg/kg).

- Note that the 'flux-form' for concentration

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\, \vec{v}) = q \nabla \cdot \vec{v}. \tag{3}$$

only takes the same form as in (1) if the flow is non-divergent $\nabla \cdot \vec{v} = 0$. *So for non-divergent flows one does not need to distinguish between density* ρ *and concentration* q *for idealized testing* (assuming $\nabla \cdot \vec{v} = 0$ in scheme).

$\Rightarrow$ For non-divergent idealized test cases the modeler is not forced to solve both (1) and (2); not forced to consider the coupling between the two!

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- Consistency: In discretized version of (1) and (2), equation (2) should reduce to (1) when $q = 1$

Why a new idealized test case?

- Need more rigorous benchmark test cases in a challenging environment to:
  - test schemes under divergent and highly deformational flow conditions

    vortices test cases: Nair and Machenhauer (2002); Nair and Jablonowski (2008)
  - test schemes on new unstructured spherical grids
  - test static and dynamic mesh refinement algorithms
  - test trajectory algorithms (semi-Lagrangian or Lagrangian methods) when parcel trajectories are not great-circle arcs

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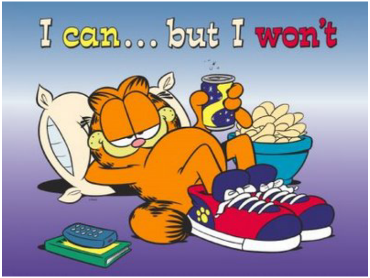


- **test case has to be simple to implement otherwise (most) people will not use it!**

Test case strategy

- Very hard to derive complex flows that have analytical solutions
- So we follow ideas developed by LeVeque (1996) and use a time-reversing flow field:

$$\vec{v}(\lambda, \theta, t) = \widetilde{\vec{v}}(\lambda, \theta) \cos\left(\frac{\pi t}{T}\right), \tag{4}$$

  where T is the period.
- Exact solution $\psi(\lambda, \theta, t = T) = \psi(\lambda, \theta, t = 0)$ initial condition
- $\Rightarrow$ can compute 'exact' error norms at $t = T$

- Upstream parcel trajectories computed using high-order Taylor Series expansions:

$$\vec{x}_d \equiv \vec{x}(t - \Delta t) = \vec{x}(t) - \Delta t \frac{d}{dt}\vec{x}(t) + \frac{(\Delta t)^2}{2!}\frac{d^2}{dt^2}\vec{x}(t) - \cdots \tag{5}$$

$$\frac{d\lambda}{dt} = \frac{u}{\cos(\theta)} \qquad \frac{d\theta}{dt} = v$$

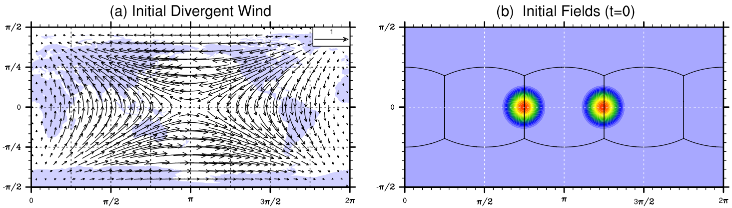
  where $\vec{v} = d\vec{x}(t)/dt$ is wind vector

- Wind components are given by:

$$u(\lambda,\theta,t) = -k\sin^2(\lambda/2)\sin(2\theta)\cos(\pi t/T) \quad (6)$$

$$v(\lambda,\theta,t) = \frac{k}{2}\sin(\lambda)\cos(\theta)\cos(\pi t/T) \quad (7)$$

where k is flow parameter



- Initial condition: $\rho = 1$ & q = smooth Cosine bell or slotted cylinder (the latter to test for monotonicity).

Simulations performed with fully 2D CSLAM

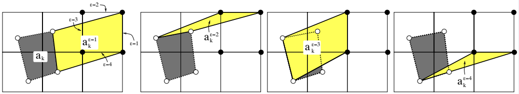
- New conservative semi-Lagrangian scheme on the cubed sphere named

  CSLAM (Conservative Semi-LAgrangian Multi-tracer transport scheme)

  - Higher-order extension of incremental remapping (Dukowicz and Baumgardner, 2000) to the cubed-sphere that supports large CFL numbers

    Lauritzen et al. (2010), Harris and Lauritzen (2010, JCP, revising), Ullrich et al. (2009)
  - Scheme is currently being extended to icosahedral grids (hexagons/triangles)

    see Mittal's talk at PDEs on the sphere workshop in August, 2010



Finite-volume flux-form of continuity equation for $\psi = \rho, \rho\, q$:

$$\int_{A_k} \psi_k^{n+1}\, dx\, dy \quad = \quad \int_{A_k} \psi_k^n\, dx\, dy - \sum_{\epsilon=1}^{4} \left[\sum_{\ell=1}^{L_k^\epsilon} s_{k\ell}^\epsilon \iint_{a_{k\ell}^\epsilon} f_\ell(x,y)\, dx\, dy \right],$$

where

- a_k^ϵ = 'flux-area' (yellow area) = area swept through face ϵ
- L_k^ϵ = number of overlap areas for a_k^ϵ; $a_{k\ell}^\epsilon = a_k^\epsilon \cap A_k$
- $s_{k\ell}^\epsilon$ = 1 for outflow and -1 for inflow.

Divergent test case 1 (Nair and Lauritzen 2010, JCP, revising)

- Animation 1: Evolution of $\rho\, q$
  - Since flow is convergent $MAX(\rho\, q)$ increases for $t \in [0, T/2]$.
- Animation 2: Evolution of q
  - q decreases for $t \in [0, T/2]$
  - q remains in interval $[0, 1]$ (as it should!)
- Animation 3: Evolution of q using slotted-cylinder initial condition using no monotone filters
  - over/under shooting
- Animation 4: Evolution of q using slotted-cylinder initial condition using monotone filter

- In this test the modeler must solve coupled system of continuity equations (tracer and air density); equation for ρ is no longer trivial as for non-divergent test cases!
- Check for consistency: $q = 1$ everywhere as initial condition should be preserved throughout simulation

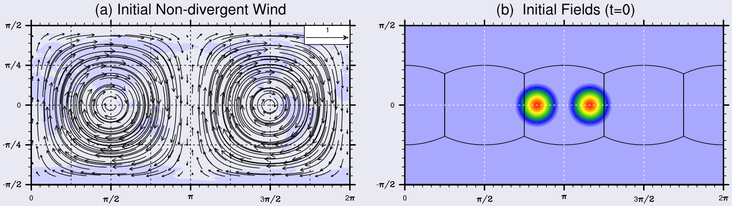
- This test is non-divergent but highly deformational.
- On top of the swirling flow we add solid-body advection $u = \cos(\theta)\, u_0$ (avoid potential cancellation of errors when flow reverses)

$$u(\lambda, \theta, t) = k \sin^2(\lambda) \sin(2\theta) \cos(\pi t/T) + \cos(\theta)\, u_0 \quad (8)$$

$$v(\lambda, \theta, t) = k \sin(2\lambda) \cos(\theta) \cos(\pi t/T) \quad (9)$$

$$\psi(\lambda, \theta, t) = k \sin^2(\lambda) \cos^2(\theta) \cos(\pi t/T), \quad (10)$$

where ψ is the stream function: $u = -\frac{\partial \psi}{\partial \theta}, v = \frac{1}{\cos\theta} \frac{\partial \psi}{\partial \lambda}$



Non-divergent test case 1 (Nair and Lauritzen 2010, JCP, revising)

- Animation 1: Evolution of q.
  Cosine bell initial condition for q ($\rho = 1$). Note that the analytic solution is $\rho(t) = 1$, however, schemes will, in general, not preserve a constant ρ unless modelers use the stream-function to make sure the numerical divergence is zero (would not be the case in a 'real' model setup when evaluating $\vec{v}$ at grid-points!).
- Animation 2: Evolution of q for slotted cylinder.
- Animation 2: Evolution of q for slotted cylinder with monotone filter.

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- Smooth and relatively well-resolved initial conditions deform into thin filament.
- Thin filaments are transported as solid-bodies near $t = T/2$

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- Thin filaments are transported as solid-bodies near $t = T/2$

- We formulated two more test cases; one more divergent and non-divergent test case, respectively.
- Test cases are still being 'fine-tuned' for the revised manuscript

Discussion

- Other configurations (relevant for chemistry) to check:
  - if linear correlations are maintained with monotone schemes (Lin and Rood, 1996)

$$q_1 = \alpha + \beta\, q_2 \tag{11}$$

  - how well relative correlations are maintained for an ‘arbitrary’ pair of concentration profiles (non-linear correlations, Thuburn and McIntyre 1997)

$$q_1 = f(q_2), \tag{12}$$

    where f is a non-linear function
- Use more complex background (ρ) distributions

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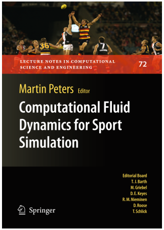
- Time will show if these test cases discriminate more between schemes as compared to other test cases.
- For CSLAM development we found moving vortex test case (Nair and Jablonowski, 2008) much more discriminating than solid-body advection test.
- These test cases have already helped people (including myself) finding bugs in the code that did not appear for other test cases (solid-body advection, static and moving vortices).

NCAR workshop

- We plan to organize a short working workshop at NCAR; tentatively March 2011 (Lauritzen, Nair, Jablonowski, Taylor, Skamarock, ...)
  - Participants must bring solutions!
  - A draft test case setup will be formulated soon (goal: get at accuracy versus cost, gradient preservation, accuracy and cost of filters/limiters, ...)
  - Input/comments are very welcome!

Lecture Notes in Computational Science and Engineering

- Springer book entitled 'Numerical Techniques for Global Atmospheric Models' based on the lectures given at the 2008 NCAR ASP (Advance Study Program) Summer Colloquium.
- Editors: P.H. Lauritzen, C. Jablonowski, M.A. Taylor and R.D. Nair
- 16 Chapters; authors include J.Thuburn, J.Tribbia, D.Durran, T.Ringler, W.Skamarock, R.Rood, J.Dennis, Editors, ... Foreword by D. Randall
- Publication date: Later this year
- More details at: http://www.cgd.ucar.edu/cms/pel/colloquium.html and http://www.cgd.ucar.edu/cms/pel/lncse.html



This book surveys recent developments in numerical techniques for global atmospheric models. It is based upon a collection of lectures prepared by leading experts in the field. The chapters reveal the multitude of steps that determine the global atmospheric model design. They encompass the choice of the equation set, computational grids on the sphere, horizontal and vertical discretizations, time integration methods, filtering and diffusion mechanisms, conservation properties, tracer transport, and considerations for designing models for massively parallel computers. A reader interested in applied numerical methods but also the many facets of atmospheric modeling should find this book of particular relevance.

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Other test cases