A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLAM) on the Cubed-Sphere

Peter H. Lauritzen [a]*, Ramachandran D. Nair* [b,*]*, Paul A. Ullrich* [c]

[a] **pel@ucar.edu**, *Climate and Global Dynamics Division, National Center for Atmospheric Research (NCAR), Boulder, Colorado, USA.*

[b] **rnair@ucar.edu**, *Institute for Mathematics Applied to Geosciences, National Center for Atmospheric Research (NCAR), Boulder, Colorado, USA.*

[c] **paullric@umich.edu**, *Department of Atmospheric, Oceanic and Space Sciences, University of Michigan, Ann Arbor, Michigan, USA.*

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Problem formulation

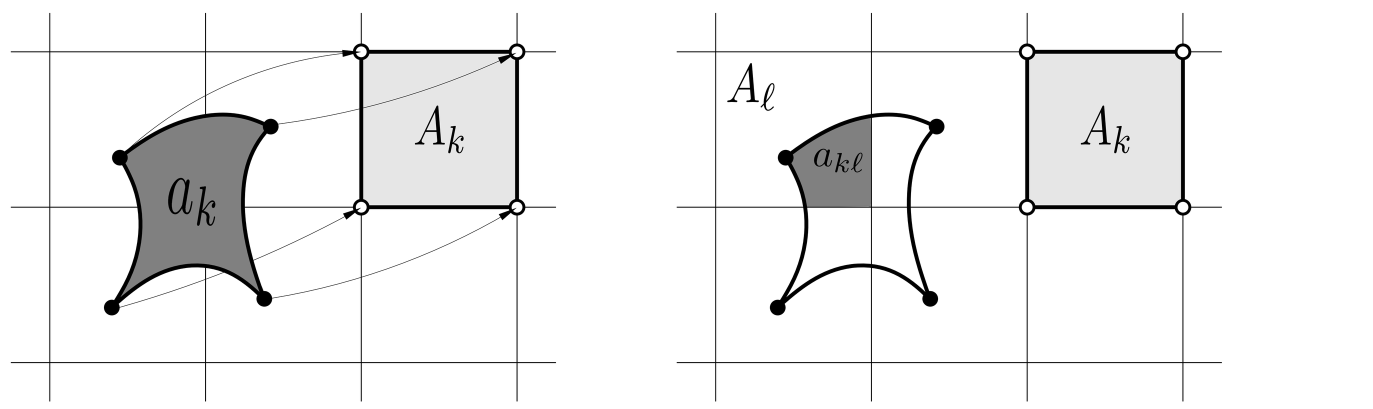
Consider two-dimensional transport equation (no sources/sinks):

$$\frac{d}{dt}\int_{A(t)} \psi \, dA = 0, \tag{1}$$

where ψ is the density, dA is the element area, and the integration is over a arbitrary Lagrangian area $A(t)$ at time t, that is, an area that moves with the flow with no flux through its boundaries. A temporal discretization of (1) reads

$$\int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA, \tag{2}$$

where Δt is time-step size. In an upstream semi-Lagrangian method $A(t+\Delta t)$ is a static grid cell (*arrival* cell) and $A(t)$ is a deformed (*departure*) cell.



Notation:

- Denote the (regular) arrival cell A_k and the corresponding departure cell a_k, where $k = 1, .., N$ and N is the total number of cells in the domain Ω. Their respective areas are denoted ΔA_k and δa_k.
- Let $a_{k\ell}$ be the non-empty overlap region between departure cell a_k and grid cell A_ℓ such that

$$a_{k\ell} = a_k \cap A_\ell, \quad a_{k\ell} \neq \emptyset; \quad \ell = 1, .., L_k, \text{ and } 1 \leq L_k \leq N.$$

- The sub-grid-scale reconstruction in cell ℓ is denoted $f_\ell(x, y)$:

$$f_\ell(x,y) = \sum_{i+j \leq 2} c_\ell^{(i,j)} x^i y^j, \quad i, j \in \{0, 1, 2\}, \tag{3}$$

where $c_\ell^{(i,j)}$ are derived coefficients ensuring mass-conservation.

Continuous scheme:

The semi-Lagrangian finite-volume version of the transport equation (1) can be written as follows:

$$\overline{\psi}_k^{n+1} \Delta A_k = \overline{\psi_k^*}^n \, \delta a_k \tag{4}$$

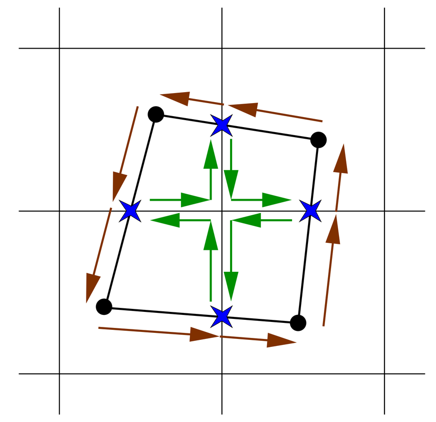
where $\overline{\psi}_k^{n+1}$ is the average tracer density in cell k at time-level $n+1$, and $\overline{\psi_k^*}^n$ is the average density in the departure cell

$$\overline{\psi_k^*}^n = \frac{1}{\delta a_k} \sum_{\ell=1}^{L_k} \iint_{a_{k\ell}} f_\ell(x,y) \, dx\, dy. \tag{5}$$

- **Given the overlap areas $a_{k\ell}$ the transport problem is effectively reduced to a remapping problem.**

Discretization in Cartesian geometry

- The departure cell a_k is defined by connecting the departure points (cell vertices) with straight lines.



We convert area integrals in (5) into line-integrals by applying the Gauss-Green theorem (Dukowicz 1984):

$$\iint_{a_{k\ell}} f_\ell(x,y) \, dx\, dy = \oint_{\partial a_{k\ell}} [P\, dx + Q\, dy],$$

where $\partial a_{k\ell}$ is boundary of $a_{k\ell}$, P and Q are potentials: $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = f_\ell(x,y)$.

- Given polynomial reconstruction function $f_\ell(x, y)$ in (3), the CSLAM scheme is given by

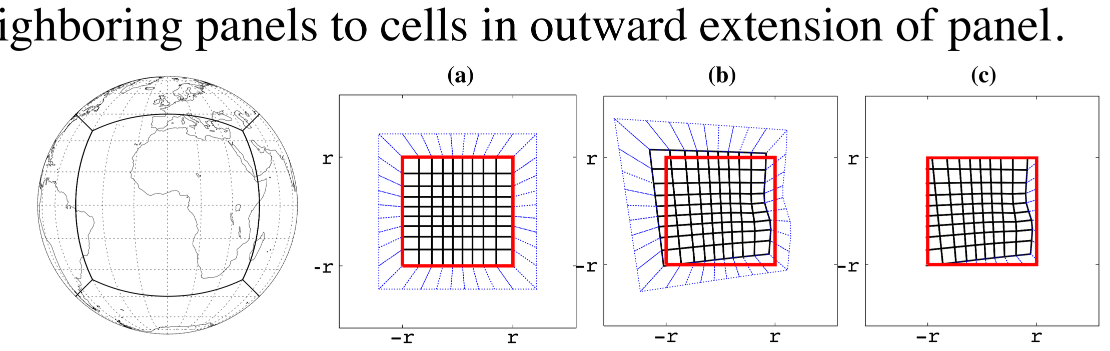
$$\boxed{\overline{\psi}_k^{n+1} \Delta A_k = \sum_{\ell=1}^{L_k} \iint_{a_{k\ell}} f_\ell(x,y) \, dx\, dy = \sum_{\ell=1}^{L_k} \left[\sum_{i+j \leq 2} c_\ell^{(i,j)} w_{k\ell}^{(i,j)} \right],} \tag{6}$$

where $w_{k\ell}^{(i,j)}$ are functions of the coordinate locations of the vertices of $a_{k\ell}$.

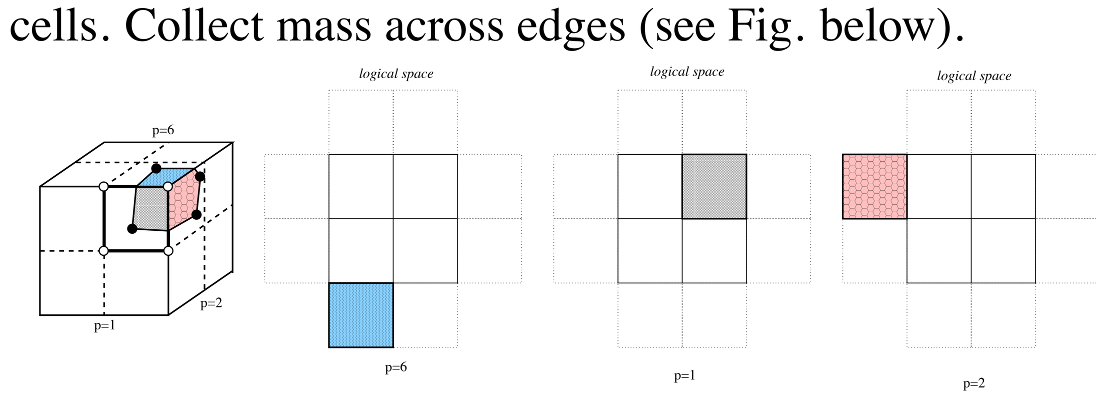
- Note: Integration of polynomials is **exact**.
- **Note: Separation of weights $w_{k\ell}^{(i,j)}$ from reconstruction coefficients $c_\ell^{(i,j)}$:**
  - *Once the weights have been computed they can be reused for the integral of each additional tracer distribution.*

Discretization in cubed-sphere geometry

- Consider cubed-sphere grids resulting from equi-angular gnomonic (central) projection. Note that any straight line on gnomonic projection corresponds to great-circle arc on sphere.
- Integrate polynomials with Gauss-Green's theorem on cube as in (6):
  - Line-integrals along coordinate lines computed exactly (Ullrich et al. 2009).
  - Line-integrals along arbitrary straight lines are approximated using Gaussian quadrature.
- Biquadratic reconstruction functions; halo values are interpolated (4^{th}-order) from neighboring panels to cells in outward extension of panel.



- Algorithm: For each panel extend panel with halo cells (Fig.a), compute departure cells (Fig.b), limit departure cells to panel (Fig.c) and compute mass in those cells. Collect mass across edges (see Fig. below).



Results

Test 1: Solid-body advection of cosine bell across cube corners (Fig.a):

- How many Gaussian quadrature points are sufficient? 2 (see Table below).
- Is the cross-term xy in the reconstruction function important? **YES**!

Scheme	number of Gauss quadrature points	l_1	l_2	l_∞
CSLAM*	**2**	**0.0949**	**0.0536**	**0.0288**
CSLAM	**2**	**0.0764**	**0.0414**	**0.0254**
CSLAM	3	0.0765	0.0414	0.0255
CSLAM	4	0.0765	0.0414	0.0255
CSLAM	5	0.0765	0.0414	0.0255

* = no cross term in $f_\ell(x,y)$

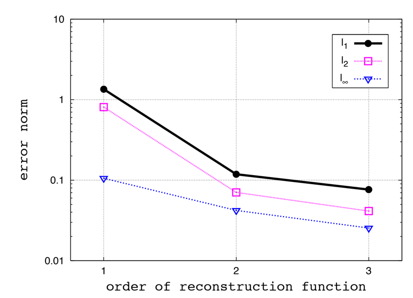
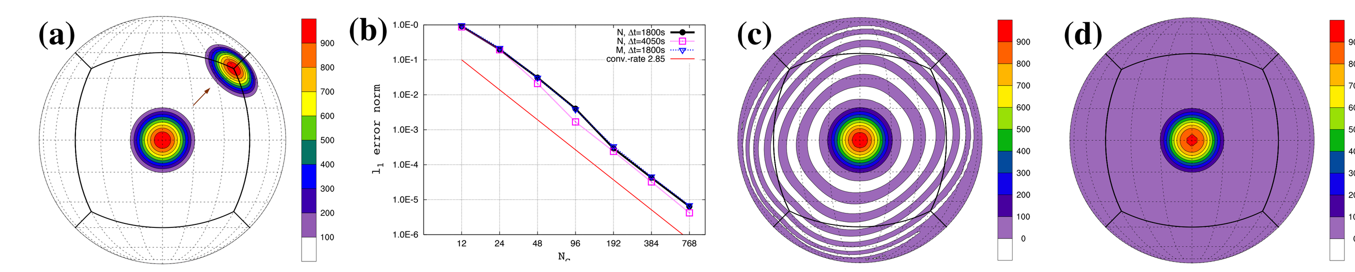


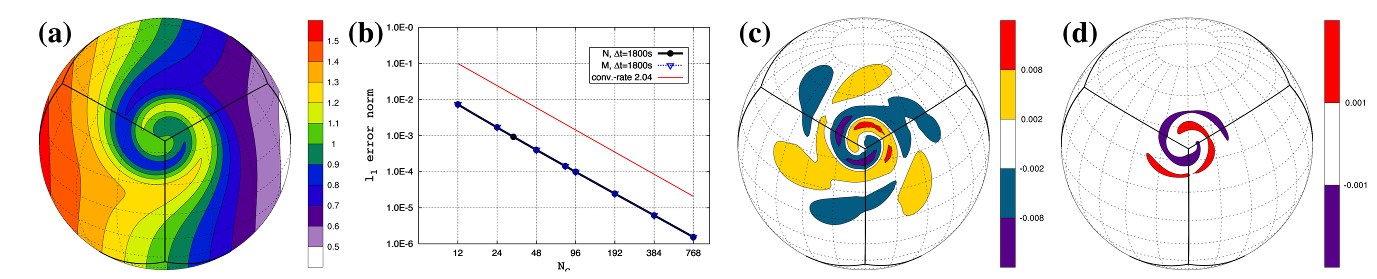
Figure shows convergence with degree of reconstruction polynomial.

- (b) Convergence plot for different Δt's for one revolution (N_c^2= number of cells on a panel). (c) unlimited (N) solution and (d) monotone solution (M).

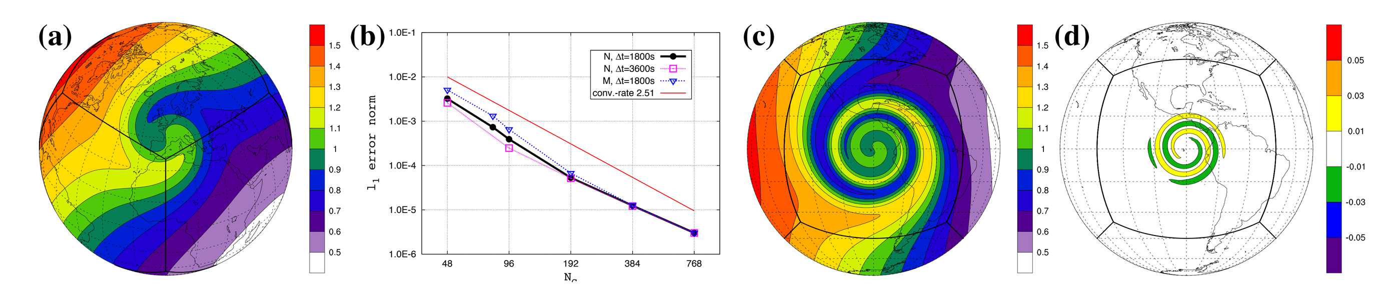


Test 2: Deformational flow test cases:

- ***Static vortices:*** Center vortices around cube corner (Fig. a). (b) Convergence plots. (c) and (d) are differences between analytic and CSLAM solution at day 6 at resolutions $N_c = 32$ and $N_c = 80$, respectively.



- ***Moving vortices*** test case recently introduced by Nair and Jablonowski (2008): (a) solution after 1/4 revolution, (b) convergence plot, (c) and (d) CSLAM and difference after one revolution at $N_c = 80$.



$\hookrightarrow$ In all tests: Better than 2^{nd}-order convergence (Lauritzen et al. 2009).
$\hookrightarrow$ CSLAM is approximately one order of magnitude more accurate than Putman and Lin (2007) scheme.

References

Dukowicz, J., 1984: Conservative rezoning (remapping) for general quadrilateral meshes. *J. Comput. Phys.*, **54**, 411–424.

Lauritzen, P. H., R. D. Nair, and P. A. Ullrich, 2009: A conservative semi-Lagrangian multi-tracer transport scheme (cslam) on the cubed-sphere grid. *J. Comput. Phys.*, submitted.

Nair, R. D. and C. Jablonowski, 2008: Moving vortices on the sphere: A test case for horizontal advection problems. *Mon. Wea. Rev.*, **136**, 699–711.

Putman, W. M. and S.-J. Lin, 2007: Finite-volume transport on various cubed-sphere grids. *J. Comput. Phys.*, **227**, 55–78.

Ullrich, P. A., P. H. Lauritzen, and C. Jablonowski, 2009: Geometrically exact conservative remapping (GECoRe): Regular latitude-longitude and cubed-sphere grids. *Mon. Wea. Rev.*, to appear.