Implementing mixed finite elements on curved elements on the sphere

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Part of NERC/STFC/Met Office UK Gung-Ho project

7th April 2014

Affine elements

An affine element is an element that can be obtained from translation plus linear transformation of the canonical reference element.

Non-affine elements occur if we have:

- quadrilaterals on sphere,
- higher-order triangulations on the sphere,
- 3D prism mesh of spherical annulus (unless shallow atmosphere approximation is used).

Take-home message

Special care must be taken when using compatible finite element spaces with non-affine elements.

Compatible finite element spaces

$$
\begin{array}{ccccc}
H^1 & \xrightarrow{\nabla^\perp} & H(\mathrm{div}) & \xrightarrow{\nabla\cdot} & L^2 \\
\downarrow \pi_0 & & \downarrow \pi_1 & & \downarrow \pi_2 \\
\mathbb{V}^0 & \xrightarrow{\nabla^\perp} & \mathbb{V}^1 & \xrightarrow{\nabla\cdot} & \mathbb{V}^2
\end{array}
$$

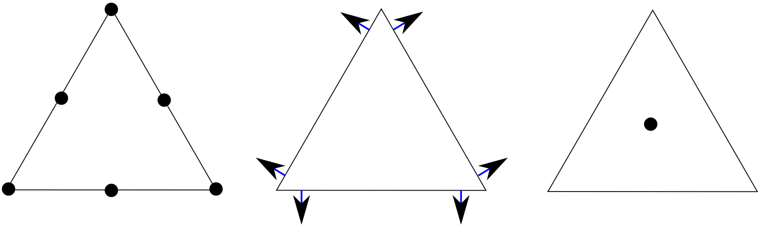
Requirements

1. $\nabla\cdot$ maps from $\mathbb{V}^1$ onto $\mathbb{V}^2$, and $\nabla^\perp$ maps from $\mathbb{V}^0$ onto kernel of $\nabla\cdot$ in $\mathbb{V}^1$.
2. Commuting, bounded surjective projections π_i exist.

Application to SWE, steady geostrophic modes, absence of spurious pressure modes, necessary conditions for absence of spurious mode branches: CJC and J. Shipton, *Mixed finite elements for numerical weather prediction*, JCP (2012).

Example FE spaces

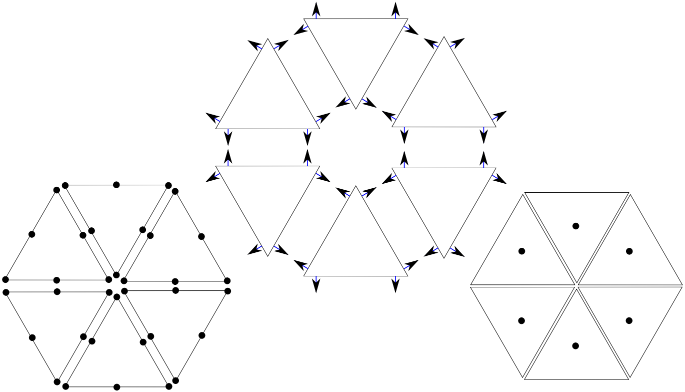
$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



Vorticity Velocity Pressure

Example FE spaces

$$\underbrace{\mathbb{V}_0 = P2}_{\text{Quadratic, Continuous}} \xrightarrow{\nabla^\perp} \underbrace{\mathbb{V}_1 = BDM1}_{\text{Linear, Continuous normals}} \xrightarrow{\nabla\cdot} \underbrace{\mathbb{V}_2 = P0}_{\text{Constant, Discontinuous}}$$



Construction strategy

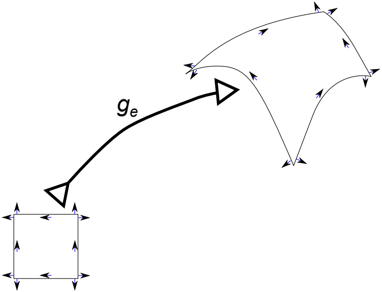
Strategy for constructing $\mathbb{V}_0, \mathbb{V}_1, \mathbb{V}_2$ on curved surfaces

1. Choose a reference element $\hat{e}$, and construct $\mathbb{V}_i(\hat{e})$, $i = 0, 1, 2$, such that:
   1. $\nabla^\perp$ maps $\mathbb{V}_0(\hat{e})$ to $\mathbb{V}_1(\hat{e})$, and
   2. $\nabla\cdot$ maps $\mathbb{V}_1(\hat{e})$ to $\mathbb{V}_2(\hat{e})$.
2. For each mesh element e, choose $g_e : \hat{e} \to e$ and find transformations $\mathbb{V}_i(\hat{e}) \to \mathbb{V}_i(e)$ such that:
   1. $\nabla^\perp$ maps $\mathbb{V}_0(e)$ to $\mathbb{V}_1(e)$,
   2. $\nabla\cdot$ maps $\mathbb{V}_1(e)$ to $\mathbb{V}_2(e)$, and
   3. interelement continuity conditions are satisfied.

For $\psi \in \mathbb{V}_0(e)$ we take $\psi \circ g_e := \psi' \in \mathbb{V}_0(\hat{e})$.

What about $\mathbb{V}_1$ and $\mathbb{V}_2$?

Construction of $\mathbb{V}_1$



$g_e : \hat{e} \to e$, $\boldsymbol{x} = g_e(\hat{\boldsymbol{x}})$.

Definition (Piola transformation)

The Piola transformation $\hat{\boldsymbol{u}} \mapsto \boldsymbol{u}$:

$$\boldsymbol{u} \circ g_e = \frac{1}{\det J} J \hat{\boldsymbol{u}}, \qquad J = \frac{\partial g_e}{\partial \hat{\boldsymbol{x}}}.$$

Properties

(1) $\int_f \hat{\phi} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s = \int_{g_e(f)} \phi \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}s$, $\phi \circ g_e = \hat{\phi}$.
Property (1) ensures correct interelement continuity.
(2) $(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}) \circ g_e = \frac{\nabla_{\hat{\boldsymbol{x}}} \cdot \hat{\boldsymbol{u}}}{\det J}$.

Implementation: M. Rognes, D. Ham, CJC and A. McRae, *Automating the solution of PDEs on the sphere and other manifolds in FeniCS* (GMDD, 2013).

Construction of $\mathbb{V}_2$

Properties of Piola transformation

(1) $\int_f \hat{\phi}\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} \mathrm{d}s = \int_{g_e(f)} \phi \boldsymbol{u} \cdot \boldsymbol{n} \mathrm{d}s$, $\phi \circ g_e = \hat{\phi}$.

(2) $(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{u}) \circ g_e = \frac{\nabla_{\hat{\boldsymbol{x}}} \cdot \hat{\boldsymbol{u}}}{\det J}$.

Property (2) then prescribes how $\mathbb{V}_2(e)$ must be constructed.

To satisfy $\boldsymbol{u}^\delta \in \mathbb{V}^1(e) \implies \nabla \cdot \boldsymbol{u}^\delta \in \mathbb{V}^2(e)$, we must have

$$\phi^\delta \circ g_e = \frac{\hat{\phi}^\delta}{\det J}, \qquad \text{for } \hat{\phi}^\delta \in \mathbb{V}^2(\hat{e}).$$

Reconstructing the mass flux

$$D_t + \nabla \cdot (\boldsymbol{u}D) = 0.$$

Choose: $D^\delta \in \mathbb{V}^2$, $\boldsymbol{u}^\delta \in \mathbb{V}^1$.

Mass flux reconstruction

For any spatial discretisation using these spaces we can find $\mathbb{F}^\delta \in \mathbb{V}^1$ such that

$$D_t^\delta + \nabla \cdot \boldsymbol{F}^\delta = 0, \text{ POINTWISE}.$$

Local construction of $\boldsymbol{F}^\delta$ depends crucially on integration by parts:

$$\int_e \phi^\delta \nabla \cdot \boldsymbol{F}^\delta \, \mathrm{d}x = -\int_e \nabla\phi^\delta \cdot \boldsymbol{F}^\delta \, \mathrm{d}x + \int_{\partial e} \phi^\delta \boldsymbol{F}^\delta \cdot \boldsymbol{n} \, \mathrm{d}s.$$

so integration must be done exactly[1].

[1] See Jemma Shipton's poster for details.

Reconstructing the mass flux (II)

$$\int_e \phi^\delta \nabla \cdot \boldsymbol{F}^\delta \, \mathrm{d}x = \int_{\hat{e}} \frac{\hat{\phi}^\delta}{\det J} \frac{\nabla_{\hat{\boldsymbol{x}}} \cdot \hat{\boldsymbol{F}}^\delta}{\det J} \det J \, \mathrm{d}\hat{x},$$
$$= \int_{\hat{e}} \hat{\phi}^\delta \frac{\nabla_{\hat{\boldsymbol{x}}} \cdot \hat{\boldsymbol{F}}^\delta}{\det J} \, \mathrm{d}\hat{x}.$$

Problem

The integrand is not polynomial and thus cannot be integrated exactly using numerical quadrature.

Solution

Choose instead that $\phi^\delta \in \mathbb{V}_2(e) \implies \phi^\delta \circ g_e = \hat{\phi}^\delta \in \mathbb{V}_2(\hat{e})$.

Secondary problem

$\boldsymbol{u}^\delta \in \mathbb{V}_1$ does not imply that $\nabla \cdot \boldsymbol{u}^\delta \in \mathbb{V}_2$ any more.

Solution

Replace $\nabla \cdot \boldsymbol{u}^\delta$ with $\pi_2 \nabla \cdot \boldsymbol{u}^\delta$.

$$
\begin{array}{ccccc}
H^1 & \xrightarrow{\nabla^\perp} & H(\mathrm{div}) & \xrightarrow{\nabla\cdot} & L^2 \\
\downarrow \pi_0 & & \downarrow \pi_1 & & \downarrow \pi_2 \\
\mathbb{V}^0 & \xrightarrow{\nabla^\perp} & \mathbb{V}^1 & \xrightarrow{\pi_2 \nabla\cdot} & \mathbb{V}^2
\end{array}
$$

This is an extension of *Bochev and Ridzal* (2008) who replaced $\nabla\cdot$ with DIV in the particular case of RT0 on quadrilaterals.

Mixed Helmholtz problem

Strong primal form:

$$\nabla^2 D - D = f.$$

Strong mixed form:

$$\boldsymbol{u} = \nabla D, \quad \nabla \cdot \boldsymbol{u} - D = f.$$

Weak mixed Helmholtz problem

Given f, find $\boldsymbol{u} \in H(\mathrm{div})$, $D \in L^2$, such that

$$\int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{u} \, \mathrm{d}x + \int_\Omega \nabla \cdot \boldsymbol{\tau} D \, \mathrm{d}x = 0, \ \forall \boldsymbol{\tau} \in H(\mathrm{div}),$$
$$-\int_\Omega v D \, \mathrm{d}x + \int_\Omega v \nabla \cdot \boldsymbol{u} \, \mathrm{d}x = \int_\Omega v f \, \mathrm{d}x, \ \forall v \in L^2.$$

Discrete mixed Helmholtz problem

Given f, find $\boldsymbol{u}^\delta \in \mathbb{V}^1$, $D^\delta \in \mathbb{V}^2$, such that

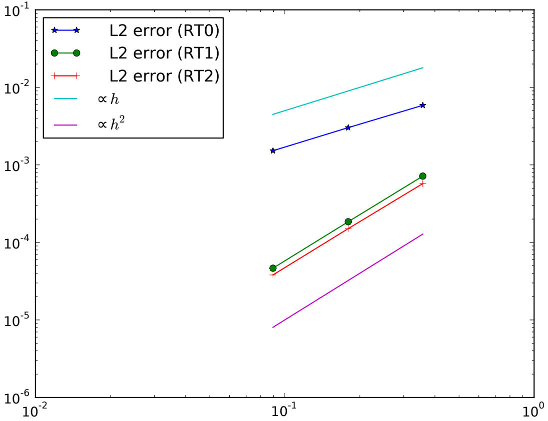
$$\int_\Omega \boldsymbol{\tau}^\delta \cdot \boldsymbol{u}^\delta \,\mathrm{d}x + \int_\Omega \nabla \cdot \boldsymbol{\tau}^\delta D^\delta \,\mathrm{d}x = 0, \ \forall \boldsymbol{\tau}^\delta \in \mathbb{V}^1,$$
$$-\int_\Omega v^\delta D^\delta \,\mathrm{d}x + \int_\Omega v^\delta \nabla \cdot \boldsymbol{u}^\delta \,\mathrm{d}x = \int_\Omega v^\delta f \,\mathrm{d}x, \ \forall v^\delta \in \mathbb{V}^2.$$

Theorem

For the conditions on V_0, V_1, V_2, described above, a unique solution D^δ exists, with $\|D - D^\delta\|_{L^2}$ converging at the optimal rate.

Unifying theorem in Arnold, Falk, Winther (Bull. Amer. Math. Soc, 2010) generalises this and collects together various results from Brezzi, Fortin, Raviart, etc.

Convergence for flat elements



Cannot achieve better than second order with flat elements.

Practical implementation

Trick

Take mapping g from flat element mesh Ω' to curved element mesh Ω, and define $\boldsymbol{u}', \boldsymbol{\tau}' \in \mathbb{V}_1(\Omega')$, ϕ', $D' \in \mathbb{V}_2(\Omega')$ via:

$$\boldsymbol{u}^\delta \circ g = \frac{J\boldsymbol{u}'}{\det J}, \quad \phi^\delta \circ g = \phi'.$$

Pullback implies that

$$\int_\Omega \phi^\delta \nabla \cdot \boldsymbol{u}^\delta \,\mathrm{d}x = \int_{\Omega'} \phi' \nabla \cdot \boldsymbol{u}' \,\mathrm{d}x'.$$

On the flat element mesh Ω', equations are:
Given f, find $\boldsymbol{u}' \in \mathbb{V}^1$, $D' \in \mathbb{V}^2(\Omega')$, such that

$$\int_{\Omega'} (J\boldsymbol{\tau}') \cdot (J\boldsymbol{u}') \frac{\mathrm{d}x}{\det J} + \int_{\Omega'} \nabla \cdot \boldsymbol{\tau}' D' \,\mathrm{d}x = 0, \ \forall \boldsymbol{\tau}' \in \mathbb{V}^1(\Omega'),$$

$$- \int_{\Omega'} v' D' \det J \,\mathrm{d}x + \int_{\Omega'} v' \nabla \cdot \boldsymbol{u}' \,\mathrm{d}x = \int_{\Omega'} v' f \det J \,\mathrm{d}x, \ \forall v' \in \mathbb{V}^2(\Omega').$$

On the flat element mesh Ω', equations are:
Given f, find $\boldsymbol{u}' \in \mathbb{V}^1$, $D' \in \mathbb{V}^2(\Omega')$, such that

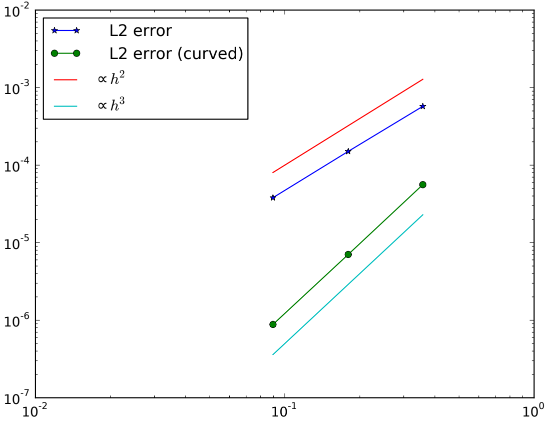
$$\int_{\Omega'} (J\boldsymbol{\tau}') \cdot (J\boldsymbol{u}') \frac{\mathrm{d}x}{\det J} + \int_{\Omega'} \nabla \cdot \boldsymbol{\tau}' D' \,\mathrm{d}x = 0, \ \forall \boldsymbol{\tau}' \in \mathbb{V}^1(\Omega'),$$

$$- \int_{\Omega'} v' D' \det J \,\mathrm{d}x + \int_{\Omega'} v' \nabla \cdot \boldsymbol{u}' \,\mathrm{d}x = \int_{\Omega'} v' f \det J \,\mathrm{d}x, \ \forall v' \in \mathbb{V}^2(\Omega').$$

Dolfin code snippet:

```
V = FunctionSpace(mesh, "RT", 3)
Q = FunctionSpace(mesh, "DG", 2)
W = MixedFunctionSpace((V, Q))
(sigma, u) = TrialFunctions(W)
(tau, v) = TestFunctions(W)
a = (inner(J*sigma, J*tau)/detJ + div(sigma)*v
     + div(tau)*u-v*u*detJ)*dx
L = g*v*detJ*dx
w = Function(W)
solve(a == L, w)
```

Convergence on curved element mesh



Third order convergence is achieved with curved elements.

Conclusions

Conclusions

- Non-affine elements are necessary to achieve higher order convergence on curved surfaces (also necessary for quadrilateral and wedge elements on sphere).
- The properties of compatible finite elements can be restored on curved elements by replacing $\nabla\cdot$ with $\pi_2\nabla\cdot$.
- Codes for flat elements can be adapted to use curved elements with minimal intervention using transformation from flat to curved elements.
- See Jemma Shipton's poster and Tom Melvin's talk for application to shallow water equations on the sphere.
- See John Thuburn's talk for testing of alternative approach using compound elements.

References:

- CJC and J. Shipton, *Mixed finite elements for numerical weather prediction*, JCP (2012).
- M. Rognes, CJC, D. Ham and A. McRae, *Automating the solution of PDEs on the sphere and other manifolds* (GMD, 2013).
- CJC and J. Thuburn, *A finite element exterior calculus framework for the rotating shallow-water equations*, (JCP, 2014).
- A. McRae and CJC, *Energy-enstrophy conserving mixed finite element schemes for the rotating shallow water equations* (QJRMS, 2014).

Projections

Definition of $\pi_1 : H(\mathrm{div}) \to \mathbb{V}_1$, $\boldsymbol{u}^\delta = \pi_1 \boldsymbol{u}$,

1. For each element edge f, $\int_f \phi^\delta \boldsymbol{u}^\delta \cdot \boldsymbol{n} \, \mathrm{d}s = \int_f \phi^\delta \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}s$, $\forall \phi \in \mathbb{V}_2$,
2. For each element e, $\int_e \nabla \phi^\delta \cdot \boldsymbol{u}^\delta \, \mathrm{d}x = \int_e \nabla \phi^\delta \cdot \boldsymbol{u} \, \mathrm{d}x$, $\forall \phi \in \mathbb{V}_2$,
3. For each element e, $\int_e \nabla^\perp \psi^\delta \cdot \boldsymbol{u}^\delta \, \mathrm{d}x = \int_e \nabla^\perp \psi^\delta \cdot \boldsymbol{u} \, \mathrm{d}x$, $\forall \psi^\delta \in \mathbb{V}_0$ with $\psi^\delta = 0$ on ∂e.

Definition of $\pi_2 : L^2 \to \mathbb{V}_2$, $h^\delta = \pi_2 h$,
$\int_e \phi^\delta h^\delta \, \mathrm{d}x = \int_e \phi^\delta h \, \mathrm{d}x$, $\forall \phi^\delta \in \mathbb{V}_2$.

Commuting property

Diagram commutes since

$$\begin{aligned}\int_e \phi^\delta \pi_2 \nabla\cdot\boldsymbol{u}\,\mathrm{d}x &= \int_e \phi^\delta \nabla\cdot\boldsymbol{u}\,\mathrm{d}x\\ &= -\int_e \nabla\phi^\delta\cdot\boldsymbol{u}\,\mathrm{d}x + \int_{\partial e}\phi^\delta\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s,\\ &= -\int_e \nabla\phi^\delta\cdot\pi_1\boldsymbol{u}\,\mathrm{d}x + \int_{\partial e}\phi^\delta\pi_1\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}s,\\ &= \int_e \phi^\delta\nabla\cdot\pi_1\boldsymbol{u}\,\mathrm{d}x, \quad \forall\phi^\delta\in\mathbb{V}_2,\end{aligned}$$

so $\pi_2\nabla\cdot\boldsymbol{u} = \nabla\cdot\pi_1\boldsymbol{u}$.

Mass flux reconstruction

... but ... $$(\nabla \cdot \boldsymbol{F}) \circ g_e = \frac{\hat{\nabla} \cdot \hat{\boldsymbol{F}}}{\det J} \notin \mathbb{V}_2(\hat{e}),$$

so we can't write $D_t + \nabla \cdot \boldsymbol{F} = 0$ pointwise!

Solution: we have

$$\int_e \phi D_t \, \mathrm{d}x + \int_e \phi \nabla \cdot \boldsymbol{F} \, \mathrm{d}x = 0.$$

Pulling back: $$\int_{\hat{e}} \hat{\phi} \hat{D}_t \det J \, \mathrm{d}\hat{x} + \int_{\hat{e}} \hat{\phi} \hat{\nabla} \cdot \hat{\boldsymbol{F}} \, \mathrm{d}\hat{x} = 0.$$

Choose $\tilde{D}_t / \det J \approx \hat{D}_t$ such that

$$\int_{\hat{e}} \hat{\phi} \hat{D}_t \det J \, \mathrm{d}\hat{x} = \int_{\hat{e}} \hat{\phi} \tilde{D}_t \, \mathrm{d}\hat{x}.$$

Then, $\tilde{D}_t + \hat{\nabla} \cdot \hat{\boldsymbol{F}} = 0$, pointwise.