Quasi-uniform grids using a spherical helix

*Takeshi Enomoto (Kyoto University)

*enomoto.takeshi.3n@kyoto-u.ac.jp

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Workshop on the Partial Differential Equations on the Sphere 2014

Quasi-uniform grids on the sphere

Saff and Kuijlaars 1997

- Chemistry: Stable molecular structure (buckminsterfullerene)
- Physics: Location of identical point charges (J. J. Thomson's problem)
- Computation: Quadrature on the sphere and computational complexity
- Botany: Distribution of pores on pollen (Tammes's problem)
- Viral morphology, crystallography etc.

Proposed approaches

- Geodesic grids (Williamson 1968; Sadourny et al. 1968)
  NB. Spring dynamics used in NICAM
  (Tomita et al. 2001; Tomita and Satoh 2004)

- Cubed sphere (Sadourny 1972; McGregor 1996)

- Reduced (Kurihara 1965; Hortal and Simmons 1991)

- Yin-Yang (Kageyama and Sato 2004; Purser 2004)

- Fibonacci (Swinbank and Purser 2006)

- Conformally mapped polyhedra (Purser and Rančić 2011)

Dissection of an icosahedron

- Division of each side of 20 equilateral triangles into n
- $N = 10n^2 + 2$ points in total
- With bisection of edges $n = 2^l$, $N = 10(2^l)^2 + 2 = 12, 42, 162, 642, 2562, 10242, ...$

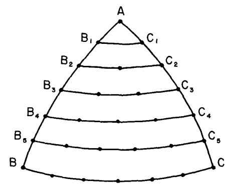
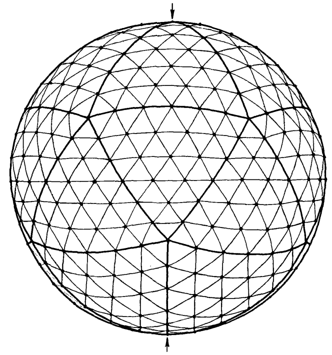
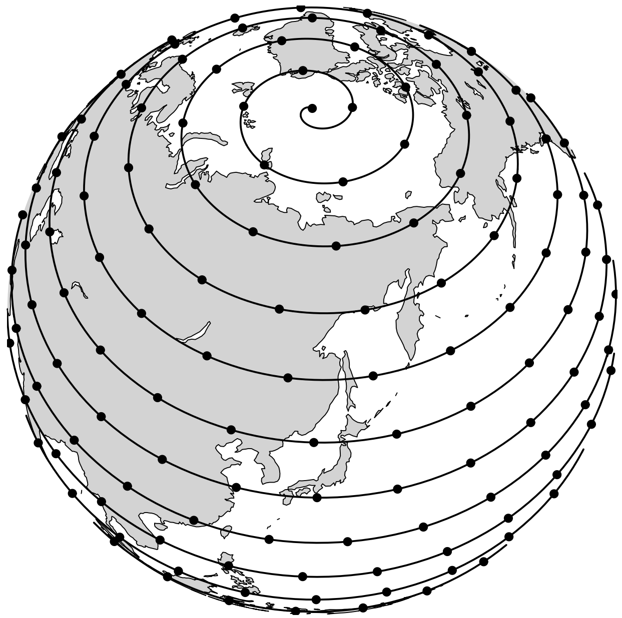


FIGURE 2.—Construction of the grid on a spherical triangle, for $n=6$.



Sadourny, Arakawa and Mintz (1968)

Spherical helix



- $\lambda = m\theta \mod 2\pi$,
  $m \equiv \mathrm{d}\lambda/\mathrm{d}\theta$ (slope)
- The length of a segment kept equal to the spacing between adjacent turns
- No limitations on the number of grids

Spherical helix for spherical SOM (self-organizing maps)

Nishio, Altaf-Ul-Amin, Kurokawa and Kanaya (2006)

$$\lambda = 2\sqrt{N}\theta \mod 2\pi \tag{1}$$

Compute the spiral length L numerically and arrange neurons at equal intervals.

With $L \approx 2m$ for large $m = 2\sqrt{N}$ the ratio between the adjacent turns and the segment length is

$$\frac{2\pi}{m}\frac{N}{2m} = \frac{N\pi}{m^2} = \frac{\pi}{4} \neq 1 \tag{2}$$

Generalized spiral points

Rakhmanov, Saff and Zhou (1994)

$$\theta_k = \arccos(h_k),\ h_k = -1 + \frac{2(k-1)}{N-1},\ 1 \le k \le N \tag{3}$$

$$\lambda_k = \left(\lambda_{k-1} + \frac{c}{\sqrt{N(1-h_k^2)}}\right) \mod 2\pi \tag{4}$$

$$c = 3.6 < \left(\frac{8}{\sqrt{3}}\right)^{1/2} = 3.809 \tag{5}$$

The best packing on the sphere

Saff and Kuijlaars (1997)

- Hexagons except for 12 pentagons in the optimal arrangement.
- The area of the hexagon with the unit distance is $\sqrt{3}/2$.
- Ignoring the pentagonal cells, assume the sphere is covered by hexagonal Dirichlet cells

$$N\frac{\sqrt{3}}{2}\delta_N^2 = 4\pi \tag{6}$$

Thus the scaling factor is $\delta_N = (8\pi/\sqrt{3})^{1/2}N^{-1/2}$

Rakhmanov et al. (1994)

$$\delta = (\lambda_k - \lambda_{k-1})\sqrt{1 - h_k} \tag{7}$$

$$= (\lambda_k - \lambda_{k-1})\sin\theta_k = \frac{c}{\sqrt{N}} \tag{8}$$

$$m = \frac{2\pi}{\delta} = \sqrt{\frac{3}{8\pi}\pi\sqrt{N}} = \sqrt{\frac{3}{2}\sqrt{\pi N}} \tag{9}$$

The ratio between the interval of the adjacent turns and the segment length is

$$\frac{2\pi}{m}\frac{N}{2m} = \frac{N\pi}{m^2} = \frac{2}{3} \neq 1 \tag{10}$$

Spherical spiral

Bauer (2000)

$$\theta_k = \arccos(h_k),\ h_k = 1 - \frac{2k-1}{N},\ 1 \le k \le N \tag{11}$$

$$\lambda = \sqrt{N\pi}\theta \mod 2\pi \tag{12}$$

With $L \approx 2m$ for large $m = \sqrt{N\pi}$ the ratio between the adjacent turns and the segment length is

$$\frac{2\pi}{m}\frac{N}{2m} = \frac{N\pi}{m^2} = 1 \tag{13}$$

Analytically exact spiral

Koay (2011)
A line element

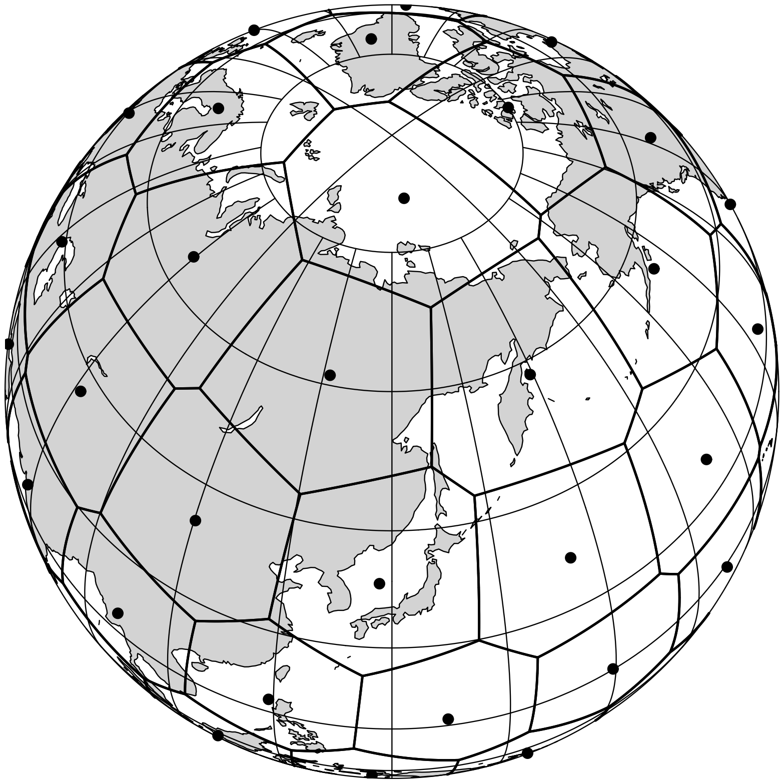
$$\mathrm{d}s = \sqrt{1 + m^2 \sin^2\theta}\mathrm{d}\theta, \; m \equiv \frac{\mathrm{d}\lambda}{\mathrm{d}\theta} \tag{14}$$

is integrated to yield

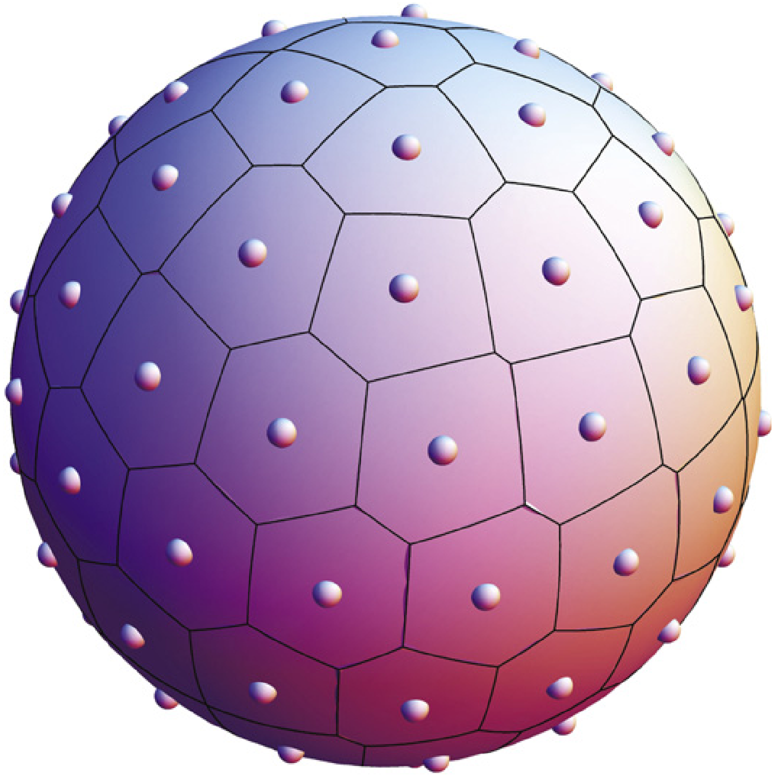
$$L(\pi) = 2E(-m^2), \; E(l) \equiv \int_0^{\pi/2} \sqrt{1 - l\sin^2\theta}\mathrm{d}\theta \tag{15}$$

$E(l)$ is the complete elliptic integral of the second kind and $L(\pi) \approx 2m$ for large m.

Bauer (2000)



Koay (2011)



Energy minimization on a sphere

The generalized energy for N points $\omega_N = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_N\}$ on the sphere

$$E(\alpha, \omega_N) \equiv \begin{cases} \displaystyle\sum_{1\le i<j\le N} \log \frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} & \text{if } \alpha = 0 \\ \displaystyle\sum_{1\le i<j\le N} |\boldsymbol{x}_i - \boldsymbol{x}_j|^{\alpha} & \text{if } \alpha \neq 0 \end{cases} \tag{16}$$

Various measures

- $\alpha = 1$: Maximization of distance $E(1, \omega_N)$
- $\alpha = 0$: Minimization of the logarithmic energy $E(0, \omega_N)$ (maximization of the product of distances).
  Logarithmic extreme points
- $\alpha = -1$: J. J. Thomson's problem.
  Minimization of energy $E(-1, \omega_N)$.
  Fekete points
- $\alpha \to \infty$: The best packing on the sphere (Tammes's problem, the hard sphere problem).
  Maximization of the smallest distance among N points.

Theoretical approximation

Rakhmanov, Saff and Zhou (1994)

$$f(-1,N) = \frac{N^2}{2} - 0.55230N^{3/2} + 0.0689N^{1/2} \tag{17}$$

$$f(0,N) = -\frac{1}{4}\log\left(\frac{4}{e}\right)N^2 - \frac{1}{4}N\log N - 0.026422N + 0.13822 \tag{18}$$

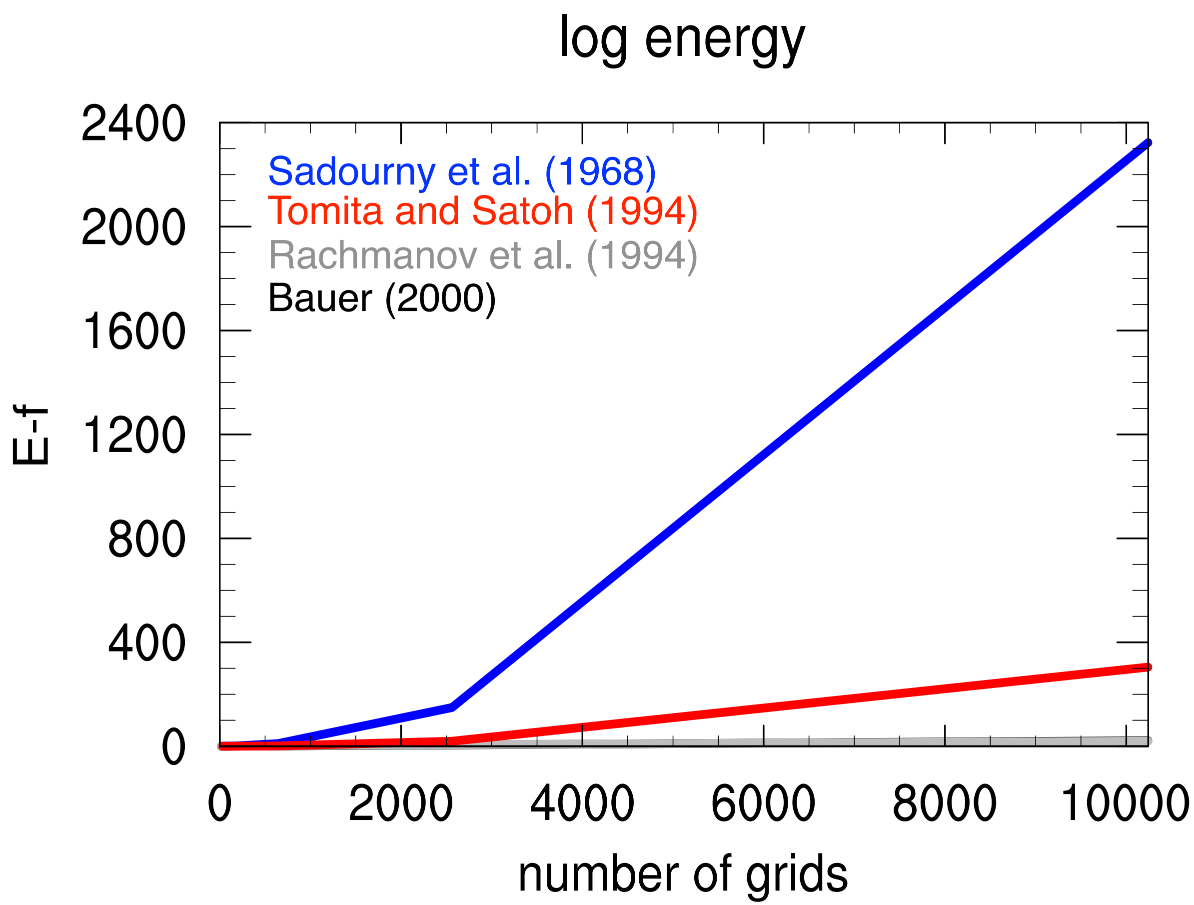
$$f(1,N) = \frac{2}{3}N^2 - 0.40096N^{1/2} - 0.188N^{-1/2} \tag{19}$$

Comparison of homogeneity

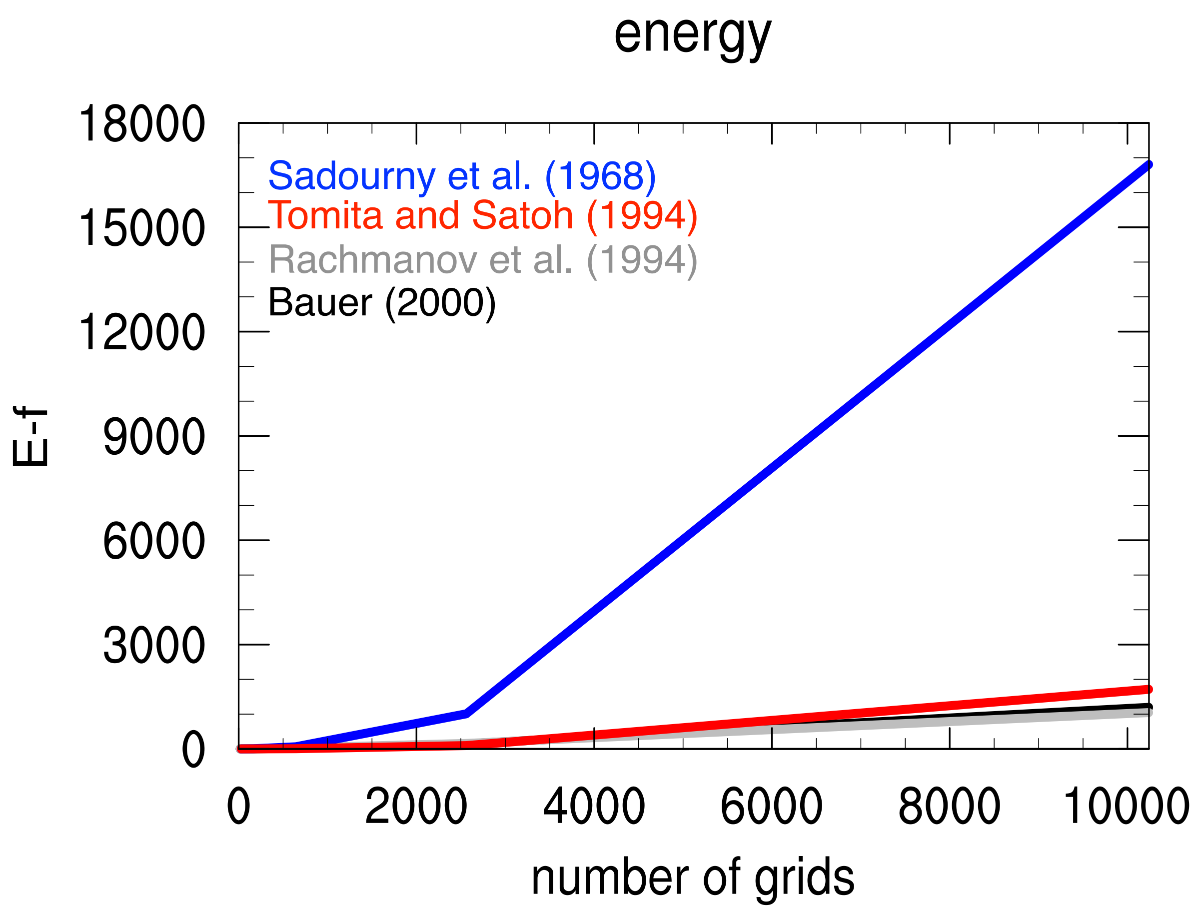
Compare norms with $N = 12, 42, 162, 642, 2562, 10242$

- Sadourny et al. (1968)
- Tomita and Satoh (1994)
- Rachmanov et al. (1994)
- Bauer (2000)

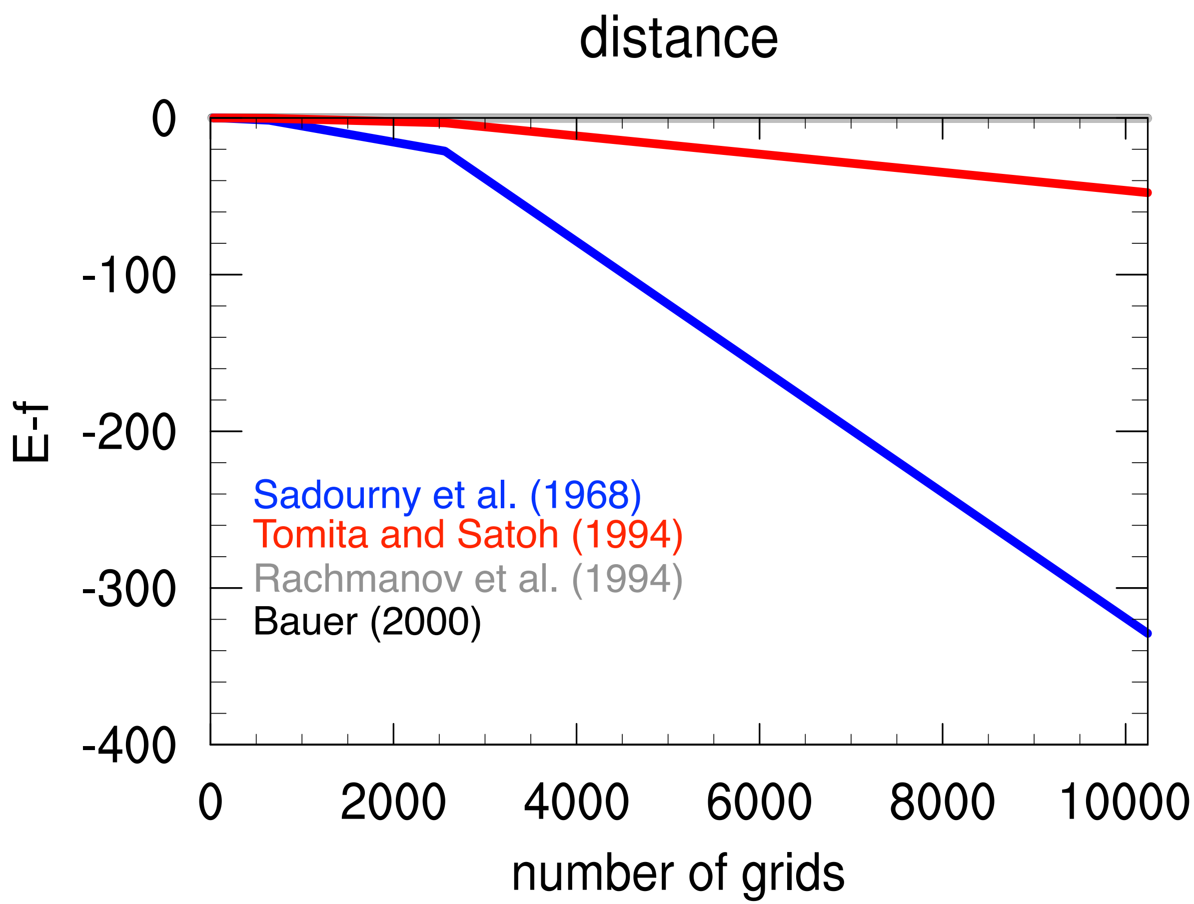
Logarithmic energy



Energy

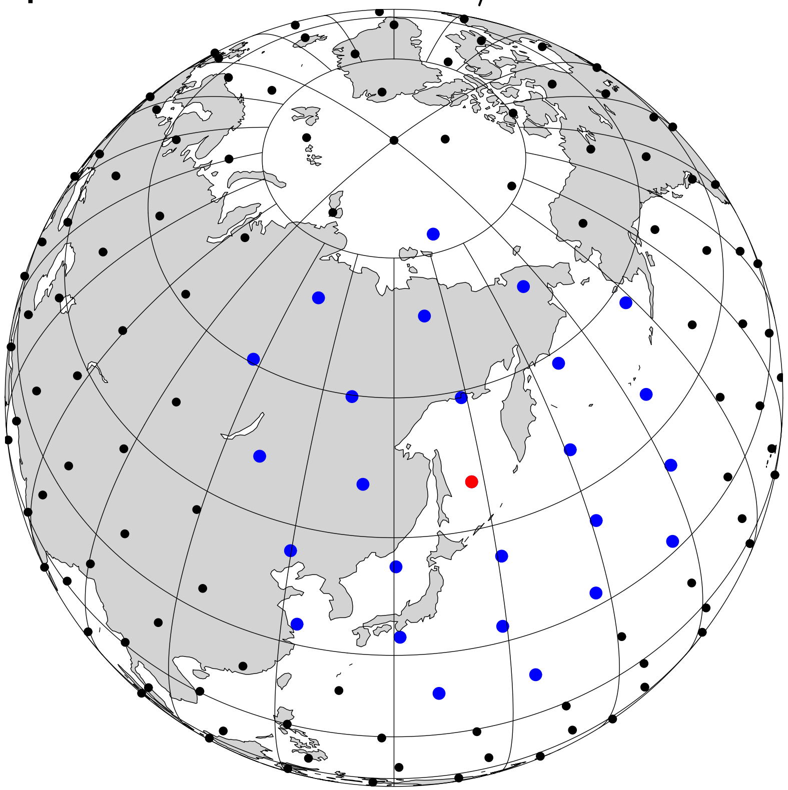

energy
18000
15000
12000
9000
6000
3000
0
E-f
0
2000
4000
6000
8000
10000
number of grids
Sadourny et al. (1968)
Tomita and Satoh (1994)
Rachmanov et al. (1994)
Bauer (2000)

Distance

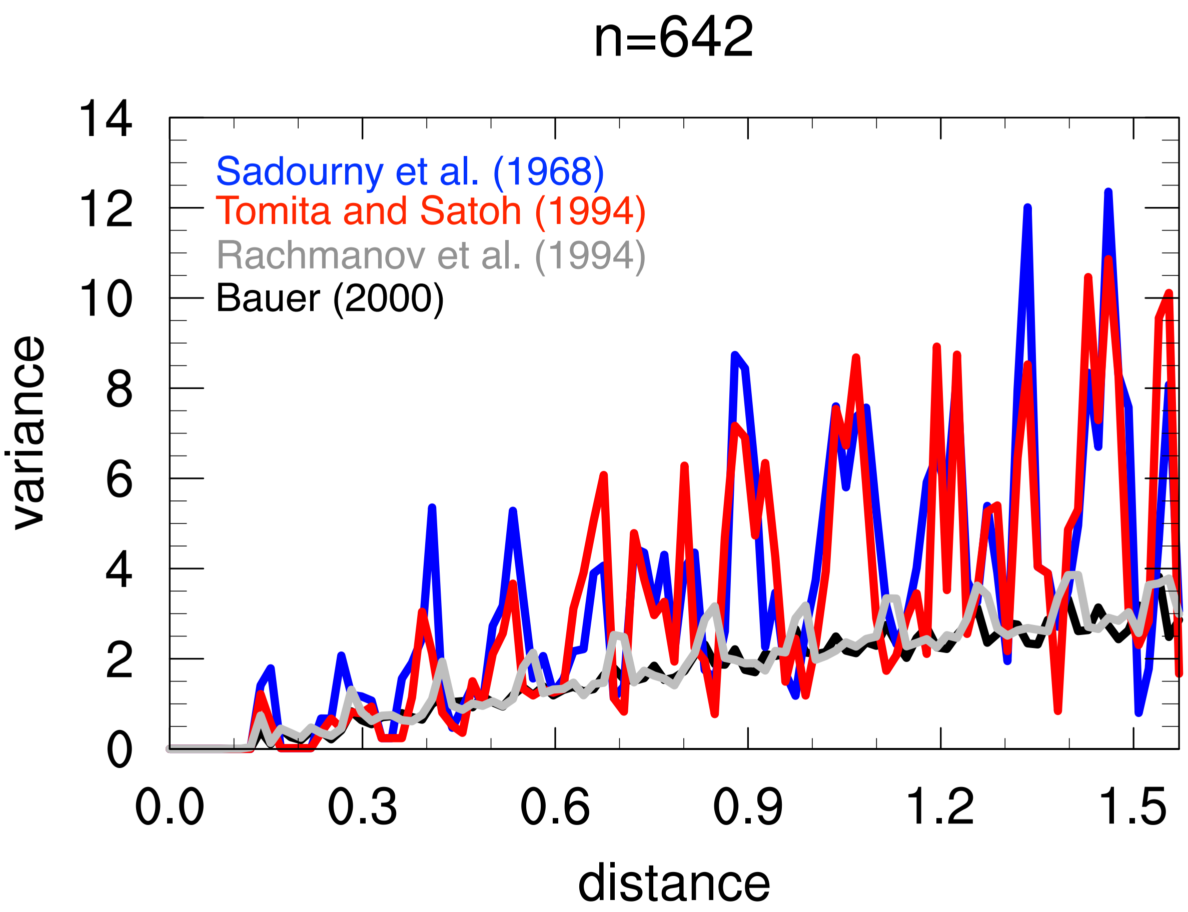

distance
0
-100
-200
-300
-400
E-f
Sadourny et al. (1968)
Tomita and Satoh (1994)
Rachmanov et al. (1994)
Bauer (2000)
0
2000
4000
6000
8000
10000
number of grids

The number of points within a radius

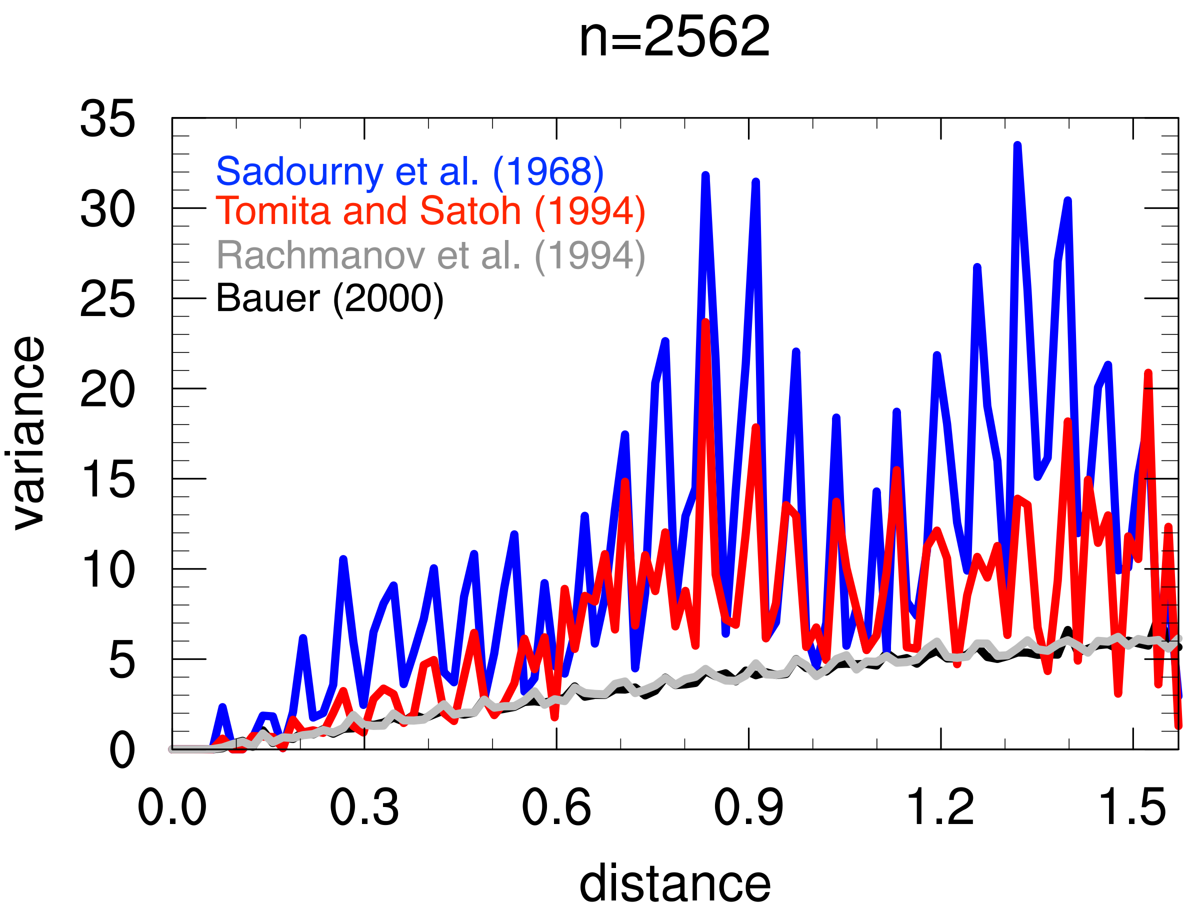
The points within $r < \pi/6$ with $N = 400$



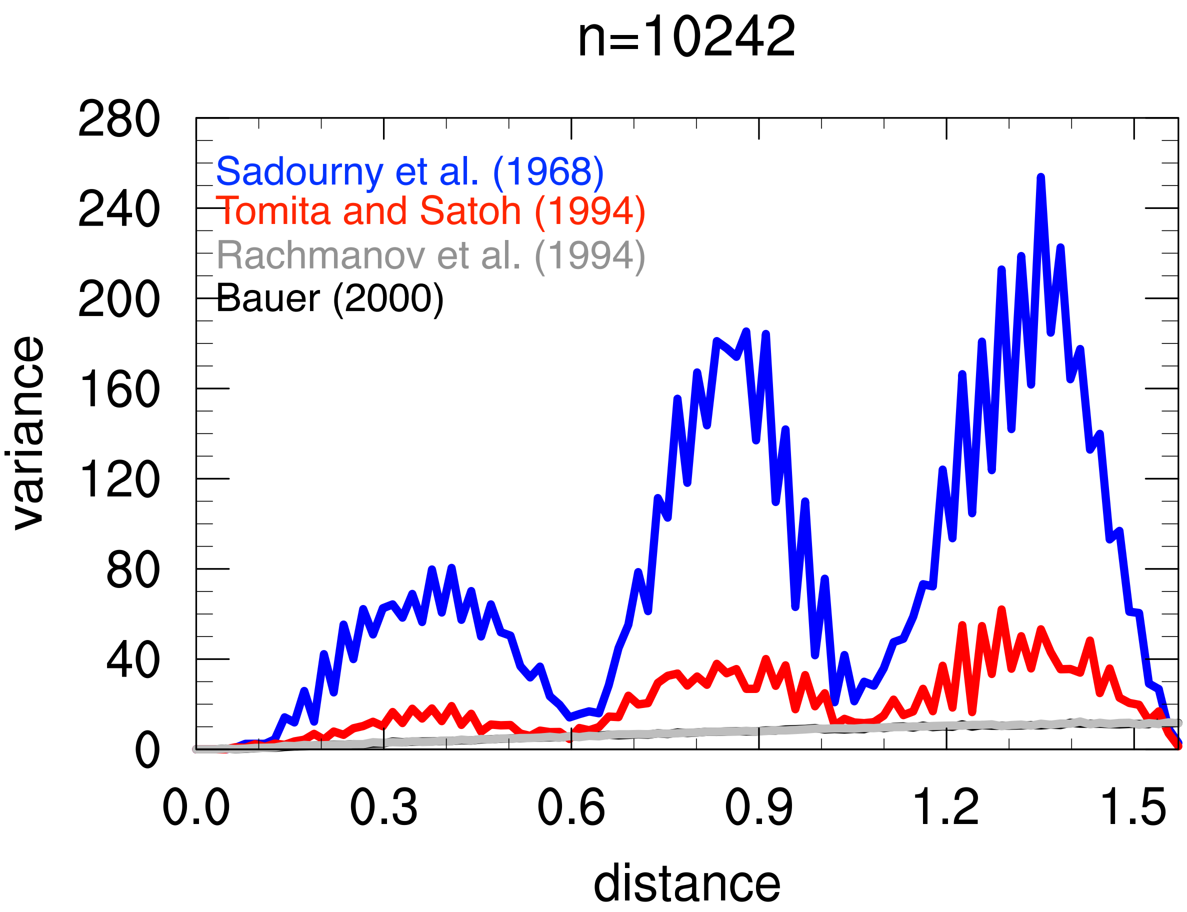
Variance $N = 642$



Variance $N = 2562$

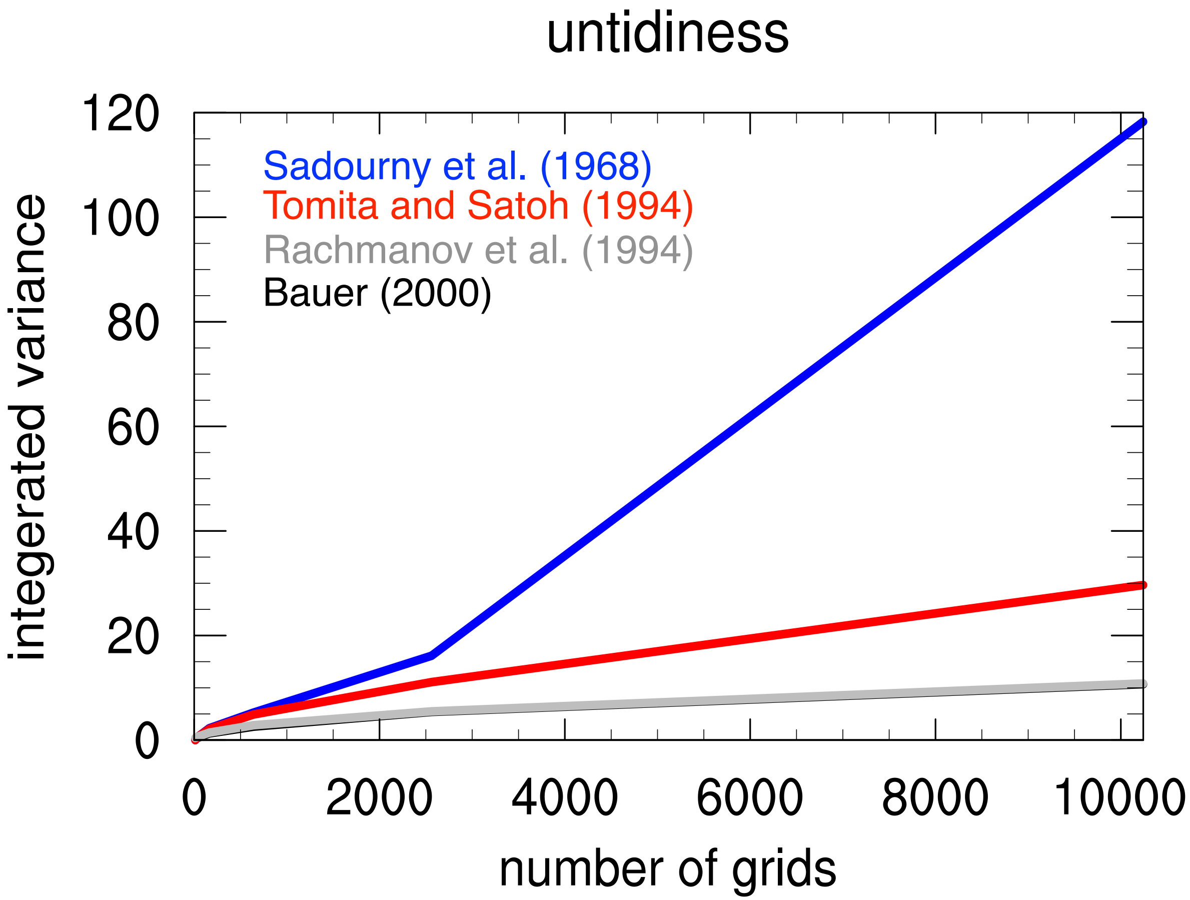


Variance $N = 10242$



"Untidiness"

Nishio et al. (2006)



Design choices

- Voronoi tessellation
- Approximate ME: quadrature with spherical harmonics
- 1D structure: interpolation. Semi-Lagrangian advection
- Weaknesses: 2D decomposition, local subdivision, ...

Summary

- Quasi-uniform grids can be easily generated with a spherical helix.
- Spherical helix grids are more uniform than geodesic grids in various measures.
- The ratio between the adjacent turns and the segment length is unity in Bauer (2000) and Koay (2011) and not in Rakhmanov et al. (1994) and Nishio et al. (1997).
- The spiral length is approximated in Bauer (2000) and computed with an iterative scheme without approximation in Koay (2011).
- Design choices remain for the use in dynamical cores