A Discretization of Deep Atmospheric Model Dynamics for NCEP Global Forecast System

Hann-Ming Henry Juang

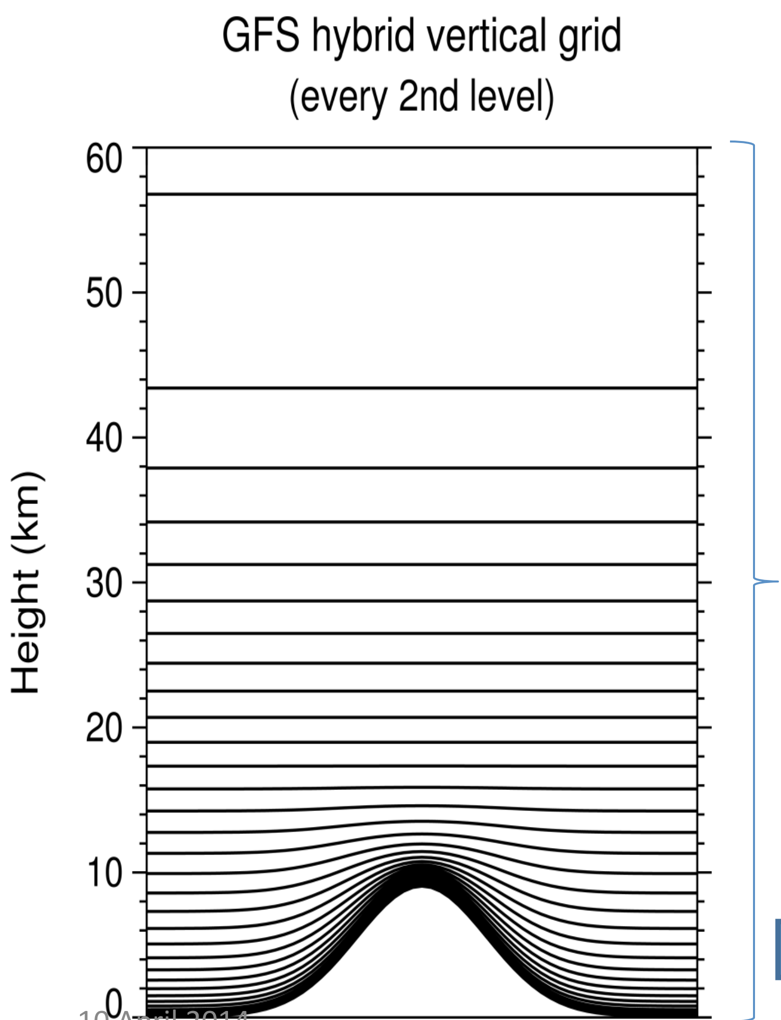
Environment Modeling Center, NOAA/NWS/NCEP, Washington, DC

Introduction

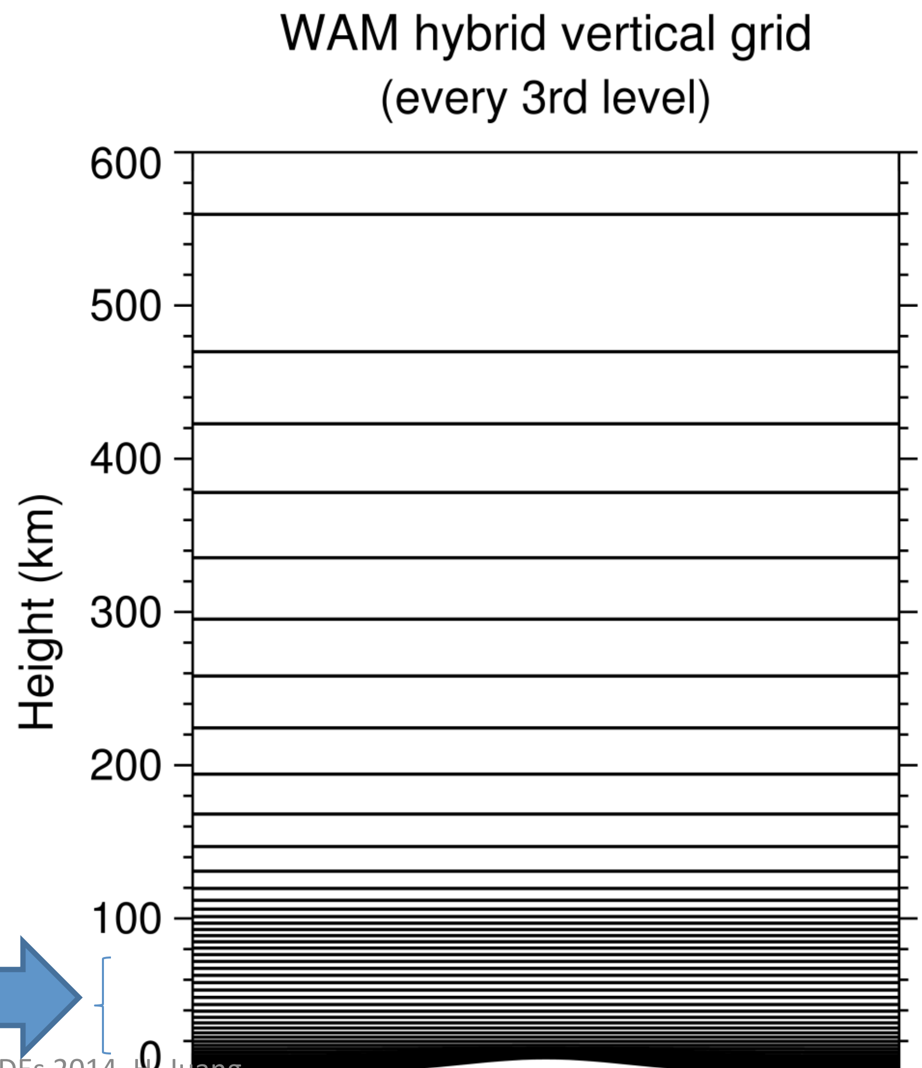
- While NCEP GFS extends its vertical domain to couple with space environmental model, we called this version of GFS as Whole Atmospheric Model (WAM). WAM is a hydrostatic system with enthalpy as thermodynamic variable (Juang 2011 MWR).
- We propose to do deep atmospheric dynamics for NCEP GFS to support WAM
- Here, we would like to re-iterate the reasons to use deep atmospheric dynamics and to illustrate the discretization of deep atmospheric dynamics for NCEP GFS.

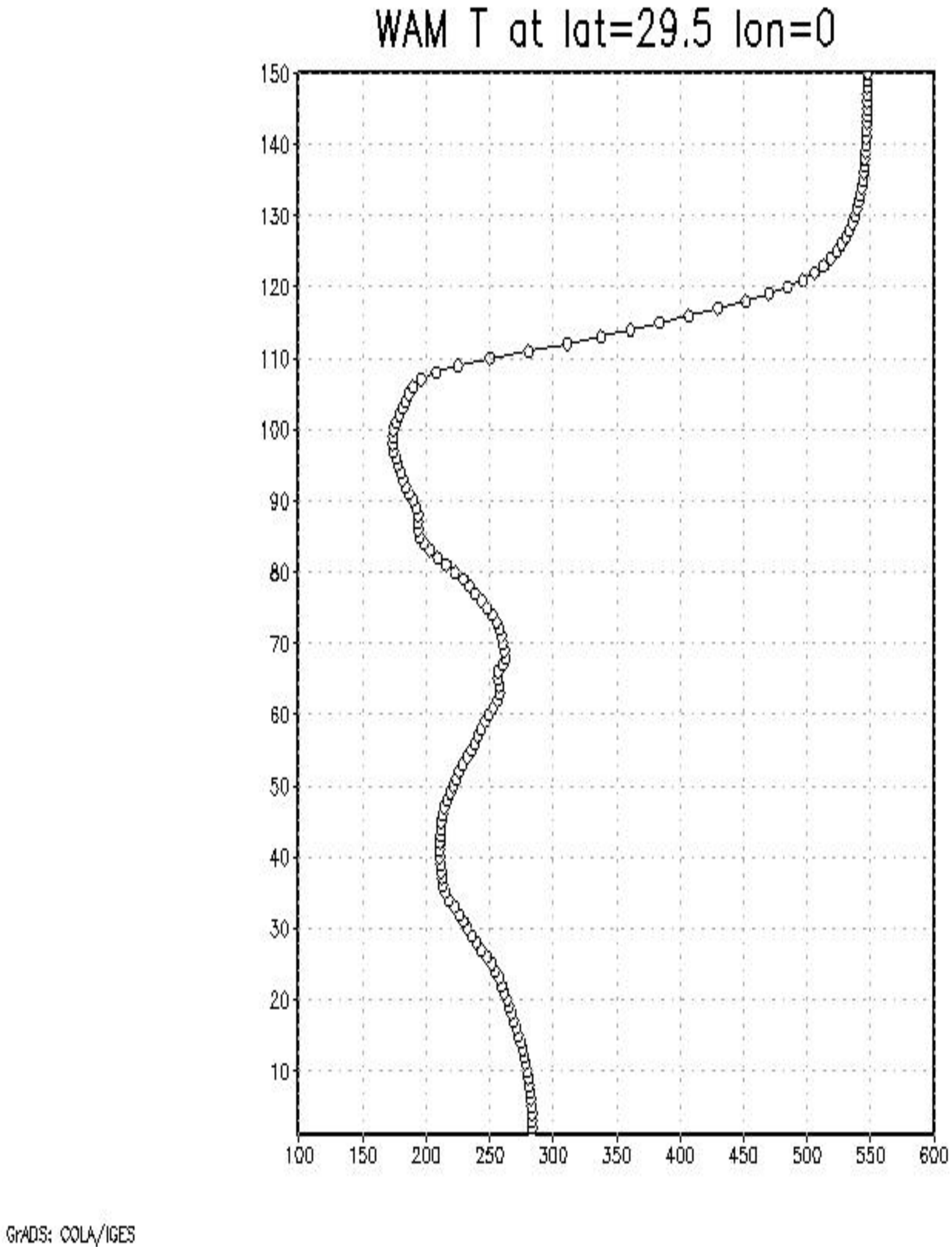
Opr GFS vs WAM

64 layers



150 layers





Example of T profile
of 150 layers

WAM uses generalized hybrid coordinate with enthalpy CpT as thermodynamics variables , where Cp is summation of each gases.

	R	Cp
O	519.674	1299.18
O2	259.837	918.096
O3	173.225	820.239
Dry air	296.803	1039.64
H2O	461.50	1846.00

Maxima wind (m/s) at NCEP GFS 150 layers WAM

```
in do_dynamics_two_loop for spdmx at kdt=      40825
spdmx(001:010)=  19.  20.  21.  23.  25.  26.  27.  28.  28.  28.
spdmx(011:020)=  28.  27.  27.  27.  27.  28.  28.  28.  29.  30.
spdmx(021:030)=  31.  33.  35.  37.  40.  42.  44.  46.  49.  53.
spdmx(031:040)=  58.  61.  63.  63.  62.  60.  55.  47.  45.  44.
spdmx(041:050)=  45.  45.  47.  49.  52.  55.  59.  62.  65.  68.
spdmx(051:060)=  72.  76.  80.  84.  87.  90.  93.  95.  97.  98.
spdmx(061:070)= 102. 110. 118. 127. 135. 143. 149. 153. 155. 152.
spdmx(071:080)= 147. 145. 142. 138. 135. 132. 130. 126. 121. 119.
spdmx(081:090)= 114. 112. 110. 106. 100.  95.  94.  90.  89.  89.
spdmx(091:100)=  87.  82.  91.  95.  99.  97. 104. 100. 111. 120.
spdmx(101:110)= 125. 133. 148. 167. 172. 164. 159. 160. 147. 124.
spdmx(111:120)= 117. 125. 133. 138. 137. 157. 183. 202. 220. 243.
spdmx(121:130)= 269. 297. 319. 338. 355. 368. 378. 386. 392. 396.
spdmx(131:140)= 399. 402. 404. 405. 406. 407. 408. 409. 410. 410.
spdmx(141:150)= 411. 412. 412. 413. 413. 414. 414. 415. 415. 418.
```

Shallow (r=a) vs Deep (r=a+z)

- Assume at z=637.12km, so r = 1.1a
- For shallow dynamic

$$u = a\cos\phi\frac{d\lambda}{dt}$$

- For deep atmosphere

$$u = r\cos\phi\frac{d\lambda}{dt}$$

- For example, ϕ=45°, u=400m/s, after one hour advection, the displacement has about 1° error in λ.

Deep atmospheric equation in height & spherical coordinates

Momentum

$$\frac{du}{dt} - \frac{uv\tan\phi}{r} + \frac{uw}{r} - (2\Omega\sin\phi)v + (2\Omega\cos\phi)w + \frac{1}{\rho}\frac{\partial p}{r\cos\phi\partial\lambda} = F_u$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{r} + \frac{vw}{r} + (2\Omega\sin\phi)u + \frac{1}{\rho}\frac{\partial p}{r\partial\phi} = F_v$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{r} - (2\Omega\cos\phi)u + \frac{1}{\rho}\frac{\partial p}{\partial r} + g = F_w$$

where

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + u\frac{\partial A}{r\cos\phi\partial\lambda} + v\frac{\partial A}{r\partial\phi} + w\frac{\partial A}{\partial r} \qquad r = a + z$$

$$u = r\cos\phi\frac{d\lambda}{dt}$$

$$v = r\frac{d\phi}{dt}$$

Gas law

$$p = \sum_n p_n = \left(\sum_n \rho_n R_n\right)T = \rho\left(\sum_n \frac{\rho_n R_n}{\rho}\right)T = \rho\left(\sum_n q_n R_n\right)T = \rho RT$$

Density

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{r\cos\phi\partial\lambda} + \frac{\partial\rho v\cos\phi}{r\cos\phi\partial\phi} + \frac{\partial\rho r^2 w}{r^2\partial r} = F_\rho$$

From IEE

$$\frac{\partial \rho e}{\partial t} + \nabla \bullet \rho e V + p \nabla \bullet V = \rho Q$$

replace

$$e = \sum_{i=1}^{N} q_i e_i = \sum_{i=1}^{N} q_i C_{V_i} T = C_V T = (C_P - R) T = h - RT$$

We have

$$\frac{\partial \rho h}{\partial t} + \nabla \bullet \rho h V - \frac{dp}{dt} = \rho Q$$

Specific value

where $C_V = \sum_{i=1}^{N} q_i C_{V_i}$ $C_P = \sum_{i=1}^{N} q_i C_{P_i}$ $R = \sum_{i=1}^{N} q_i R_i$ & $q_i = \frac{\rho_i}{\rho}$ $\rho = \sum_{n=1}^{N} \rho_n$

Combine IEE in h form with $\frac{\partial \rho}{\partial t} + \nabla \bullet \rho V = F_\rho$

We have $$\frac{dh}{dt} - \frac{\kappa h}{p} \frac{dp}{dt} = Q - \frac{h}{\rho} F_\rho$$ & we need $$\frac{dq_i}{dt} = F_{q_i}$$

where $h=CpT$ is an Enthalpy

$$\frac{du}{dt}-\frac{uv\tan\phi}{r}+\frac{uw}{r}-f_s v+f_c w+\frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)=F_u$$

$$\frac{dv}{dt}+\frac{u^2\tan\phi}{r}+\frac{vw}{r}+f_s u+\frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial\phi}-\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\phi}\right)=F_v$$

$$\frac{dw}{dt}-\frac{u^2+v^2}{r}-f_c u+\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}+g=F_w$$

$$\frac{dh}{dt}-\frac{\kappa h}{p}\frac{dp}{dt}=F_h$$

$$\frac{\partial\beta}{\partial t}+\frac{\partial\beta\dot{\lambda}}{\partial\lambda}+\frac{\partial\beta\dot{\phi}}{\partial\phi}+\frac{\partial\beta\dot{\zeta}}{\partial\zeta}=F_\rho^\beta$$

$$\frac{dq_i}{dt}=F_{q_i}$$

Deep Atmos Dyn
in generalized coordinate

Staniforth and Wood (2003)

where $\frac{d()}{dt}=\frac{\partial()}{\partial t}+\dot{\lambda}\frac{\partial()}{\partial\lambda}+\dot{\phi}\frac{\partial()}{\partial\phi}+\dot{\zeta}\frac{\partial()}{\partial\zeta}$; $\beta=\rho r^2\cos\phi\frac{\partial r}{\partial\zeta}$; $w=\frac{\partial r}{\partial t}+\dot{\lambda}\frac{\partial r}{\partial\lambda}+\dot{\phi}\frac{\partial r}{\partial\phi}+\dot{\zeta}\frac{\partial r}{\partial\zeta}$

$f_s=2\Omega\sin\phi$; $f_c=2\Omega\cos\phi$; $g=g(r)$; $p=\rho\kappa h$

Angular momentum $A = r\cos\phi\left(u + \Omega r\cos\phi\right)$

$$\frac{dA}{dt} = r\cos\phi\frac{du}{dt} + \left(u + 2\Omega r\cos\phi\right)\frac{dr\cos\phi}{dt} = r\cos\phi\frac{du}{dt} + \left(u + 2\Omega r\cos\phi\right)\left(w\cos\phi - v\sin\phi\right)$$

put $\frac{du}{dt} = \frac{uv\tan\phi}{r} - \frac{uw}{r} + f_s v - f_c w - \frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right) + F_u$ in

We have $\frac{dA}{dt} = r\cos\phi\left[F_u - \frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right)\right]$

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{u}{r\cos\phi}\frac{\partial A}{\partial\lambda} + \frac{v}{r}\frac{\partial A}{\partial\phi} + \dot{\zeta}\frac{\partial A}{\partial\zeta}$$

$$\frac{\partial\beta}{\partial t} + \frac{\partial\frac{u}{r\cos\phi}\beta}{\partial\lambda} + \frac{\partial\frac{v}{r}\beta}{\partial\phi} + \frac{\partial\dot{\zeta}\beta}{\partial\zeta} = 0$$

$$\frac{\partial\beta A}{\partial t} + \frac{\partial\frac{u}{r\cos\phi}\beta A}{\partial\lambda} + \frac{\partial\frac{v}{r}\beta A}{\partial\phi} + \frac{\partial\dot{\zeta}\beta A}{\partial\zeta} = \beta\frac{dA}{dt}$$

where $\beta = \rho r^2\cos\phi\frac{\partial r}{\partial\zeta}$

$$\frac{\partial \beta A}{\partial t}+\frac{\partial}{\partial \lambda}\left(\frac{u\beta A}{r\cos\phi}\right)+\frac{\partial}{\partial \phi}\left(\frac{v\beta A}{r}\right)+\frac{\partial \dot{\zeta}\beta A}{\partial \zeta}$$

$$=\beta r\cos\phi\left[F_u-\frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial \lambda}-\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right)\right]$$

$$=-r^2\cos\phi\frac{\partial r}{\partial \zeta}\frac{\partial p}{\partial \lambda}+r^2\cos\phi\frac{\partial p}{\partial \zeta}\frac{\partial r}{\partial \lambda}+\beta r\cos\phi F_u$$

$$=-\frac{\partial pr^2\cos\phi\frac{\partial r}{\partial \zeta}}{\partial \lambda}+p\frac{\partial r^2\cos\phi\frac{\partial r}{\partial \zeta}}{\partial \lambda}+\frac{\partial pr^2\cos\phi\frac{\partial r}{\partial \lambda}}{\partial \zeta}-p\frac{\partial r^2\cos\phi\frac{\partial r}{\partial \lambda}}{\partial \zeta}+\beta r\cos\phi F_u$$

$$=-\frac{\partial pr^2\cos\phi\frac{\partial r}{\partial \zeta}}{\partial \lambda}+\frac{\partial pr^2\cos\phi\frac{\partial r}{\partial \lambda}}{\partial \zeta}+\beta r\cos\phi F_u$$

$$\frac{\partial}{\partial t}\oiiint \rho A\,dv=\oiiint \rho r\cos\phi F_u\,dv+\oiint\left(p\frac{\partial r}{\partial \lambda}\right)_T ds-\oiint\left(p\frac{\partial r}{\partial \lambda}\right)_B ds$$

Angular momentum conserved if top p=0 or r=constant thus the bottom is the only torque.

Consider Kinetic energy from momentum equations

$$u\frac{du}{dt} - u\frac{uv\tan\phi}{r} + u\frac{uw}{r} - uf_s v + uf_c w + u\frac{\kappa h}{p}\frac{1}{r\cos\phi}\left(\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right) = 0$$

$$v\frac{dv}{dt} + v\frac{u^2\tan\phi}{r} + v\frac{vw}{r} + vf_s u + v\frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial\phi} - \frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\phi}\right) = 0$$

$$w\frac{dw}{dt} - w\frac{u^2+v^2}{r} - wf_c u + w\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r} + wg = 0$$

Sum them together, we have $K = \frac{1}{2}\left(u^2+v^2+w^2\right)$

$$\frac{\partial K}{\partial t} + \dot{\lambda}\frac{\partial K}{\partial\lambda} + \dot{\phi}\frac{\partial K}{\partial\phi} + \dot{\zeta}\frac{\partial K}{\partial\zeta} = -\frac{1}{\rho}\dot{\lambda}\left(\frac{\partial p}{\partial\lambda} - \frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\lambda}\right) - \frac{1}{\rho}\dot{\phi}\left(\frac{\partial p}{\partial\phi} - \frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\frac{\partial r}{\partial\phi}\right) - \frac{1}{\rho}w\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r} - gw$$

Combine with mass and w equations as

$$\frac{\partial\beta}{\partial t} + \frac{\partial\beta\dot{\lambda}}{\partial\lambda} + \frac{\partial\beta\dot{\phi}}{\partial\phi} + \frac{\partial\beta\dot{\zeta}}{\partial\zeta} = 0$$

and

$$w = \frac{\partial r}{\partial t} + \dot{\lambda}\frac{\partial r}{\partial\lambda} + \dot{\phi}\frac{\partial r}{\partial\phi} + \dot{\zeta}\frac{\partial r}{\partial\zeta}$$

We have

$$\frac{\partial\beta K}{\partial t} + \frac{\partial\dot{\lambda}\beta\left(K+\frac{p}{\rho}\right)}{\partial\lambda} + \frac{\partial\dot{\phi}\beta\left(K+\frac{p}{\rho}\right)}{\partial\phi} + \frac{\partial\dot{\zeta}\beta\left(K+\frac{p}{\rho}\right)}{\partial\zeta} = p\frac{\partial\frac{\beta}{\rho}\dot{\lambda}}{\partial\lambda} + p\frac{\partial\frac{\beta}{\rho}\dot{\phi}}{\partial\phi} + p\frac{\partial\frac{\beta}{\rho}\dot{\zeta}}{\partial\zeta} - \frac{\partial p\frac{\beta}{\rho}\frac{\partial r}{\partial t}\frac{\partial\zeta}{\partial r}}{\partial\zeta} + p\frac{\partial\frac{\beta}{\rho}\frac{\partial r}{\partial t}\frac{\partial\zeta}{\partial r}}{\partial\zeta} - \beta gw$$

$$= p\left(\frac{\partial\frac{\beta}{\rho}\dot{\lambda}}{\partial\lambda} + \frac{\partial\frac{\beta}{\rho}\dot{\phi}}{\partial\phi} + \frac{\partial\frac{\beta}{\rho}\dot{\zeta}}{\partial\zeta}\right) - \frac{\partial pr^2\cos\phi\frac{\partial r}{\partial t}}{\partial\zeta} + p\frac{\partial r^2\cos\phi\frac{\partial r}{\partial t}}{\partial\zeta} - \beta gw = p\left(\frac{\partial\frac{\beta}{\rho}}{\partial t} + \frac{\partial\frac{\beta}{\rho}\dot{\lambda}}{\partial\lambda} + \frac{\partial\frac{\beta}{\rho}\dot{\phi}}{\partial\phi} + \frac{\partial\frac{\beta}{\rho}\dot{\zeta}}{\partial\zeta}\right) - \frac{\partial pr^2\cos\phi\frac{\partial r}{\partial t}}{\partial\zeta} - \beta gw$$

Consider geo-potential energy

$$\int g\,dr = \Phi \qquad \text{or} \qquad g = \frac{d\Phi}{dt}$$

$$gw = \frac{d\Phi}{dr}\frac{dr}{dt} = \frac{\partial \Phi}{\partial t} + \dot{\lambda}\frac{\partial \Phi}{\partial \lambda} + \dot{\phi}\frac{\partial \Phi}{\partial \phi} + \dot{\zeta}\frac{\partial \Phi}{\partial \zeta}$$

Combine with mass equation $\frac{\partial \beta}{\partial t} + \frac{\partial \beta \dot{\lambda}}{\partial \lambda} + \frac{\partial \beta \dot{\phi}}{\partial \phi} + \frac{\partial \beta \dot{\zeta}}{\partial \zeta} = 0$

We have

$$\frac{\partial \beta \Phi}{\partial t} + \frac{\partial \dot{\lambda} \beta \Phi}{\partial \lambda} + \frac{\partial \dot{\phi} \beta \Phi}{\partial \phi} + \frac{\partial \dot{\zeta} \beta \Phi}{\partial \zeta} = \beta g w$$

Consider internal energy

$$\frac{dC_P T}{dt} - \frac{1}{\rho}\frac{dp}{dt} = \frac{dC_V T}{dt} + \frac{dRT}{dt} - \frac{1}{\rho}\frac{dp}{dt} = \frac{dC_V T}{dt} + p\frac{d\frac{1}{\rho}}{dt} = 0$$

Apply mass equations in both total derivative terms,
We have

$$\frac{\partial \beta C_V T}{\partial t} + \frac{\partial \dot{\lambda} \beta C_V T}{\partial \lambda} + \frac{\partial \dot{\phi} \beta C_V T}{\partial \phi} + \frac{\partial \dot{\zeta} \beta C_V T}{\partial \zeta} = -p\left[\frac{\partial}{\partial t}\left(\frac{\beta}{\rho}\right) + \frac{\partial}{\partial \lambda}\left(\dot{\lambda}\frac{\beta}{\rho}\right) + \frac{\partial}{\partial \phi}\left(\dot{\phi}\frac{\beta}{\rho}\right) + \frac{\partial}{\partial \zeta}\left(\dot{\zeta}\frac{\beta}{\rho}\right)\right]$$

Combine previous three red enclosed equations

$$\frac{\partial \beta K}{\partial t}+\frac{\partial \dot{\lambda}\beta\left(K+\frac{p}{\rho}\right)}{\partial \lambda}+\frac{\partial \dot{\phi}\beta\left(K+\frac{p}{\rho}\right)}{\partial \phi}+\frac{\partial \dot{\zeta}\beta\left(K+\frac{p}{\rho}\right)}{\partial r}=p\left(\frac{\partial \frac{\beta}{\rho}}{\partial t}+\frac{\partial \frac{\beta}{\rho}\dot{\lambda}}{\partial \lambda}+\frac{\partial \frac{\beta}{\rho}\dot{\phi}}{\partial \phi}+\frac{\partial \frac{\beta}{\rho}\dot{\zeta}}{\partial \zeta}\right)-\frac{\partial pr^2\cos\phi\frac{\partial r}{\partial t}}{\partial \zeta}-\beta g w$$

$$\frac{\partial \beta\Phi}{\partial t}+\frac{\partial \dot{\lambda}\beta\Phi}{\partial \lambda}+\frac{\partial \dot{\phi}\beta\Phi}{\partial \phi}+\frac{\partial \dot{\zeta}\beta\Phi}{\partial \zeta}=\beta g w$$

$$\frac{\partial \beta C_V T}{\partial t}+\frac{\partial \dot{\lambda}\beta C_V T}{\partial \lambda}+\frac{\partial \dot{\phi}\beta C_V T}{\partial \phi}+\frac{\partial \dot{\zeta}\beta C_V T}{\partial \zeta}=-p\left[\frac{\partial}{\partial t}\left(\frac{\beta}{\rho}\right)+\frac{\partial}{\partial \lambda}\left(\dot{\lambda}\frac{\beta}{\rho}\right)+\frac{\partial}{\partial \phi}\left(\dot{\phi}\frac{\beta}{\rho}\right)+\frac{\partial}{\partial \zeta}\left(\dot{\zeta}\frac{\beta}{\rho}\right)\right]$$

Integral globally with all BC, include $p_{\zeta_T}=0$ $\left(\frac{\partial r}{\partial t}\right)_{\zeta_B}=0$

$$\iiint\left(\frac{\partial \beta K}{\partial t}+\frac{\partial \beta C_V T}{\partial t}+\frac{\partial \beta\Phi}{\partial t}\right)d\zeta d\lambda d\phi=-\iint\left[\left(pr^2\cos\phi\frac{\partial r}{\partial t}\right)_{\zeta_T}-\left(pr^2\cos\phi\frac{\partial r}{\partial t}\right)_{\zeta_B}\right]d\lambda d\phi$$

and BC give $$\frac{\partial}{\partial t}\iiint\beta\left(K+C_V T+\Phi\right)d\zeta d\lambda d\phi=\frac{\partial}{\partial t}\iiint\rho\left(K+C_V T+\Phi\right)dv=0$$

Total energy conserved

Deep Atmos vs non-Hydro

- From Deep atmosphere, we require r changes with time, thus we need dw/dt equation
- And we need full curvature and Coriolis force terms to satisfy conservation
- Thus, based on conservation requirement, **a deep atmospheric dynamic is a non-hydrostatic dynamic**. **A non-hydrostatic dynamics can be shallow or deep atmospheric dynamics**.
- Both r and vertical components of curvature and Coriolis force should be considered in deep atmosphere; and should not be considered in shallow atmosphere.

$$\frac{du^*}{dt}+\frac{u^* w}{r} \quad -f_s v^* \quad +f_c^* w \qquad +\frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \lambda}-\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \lambda}\right) \quad = F_u$$

$$\frac{dv^*}{dt}+\frac{v^* w}{r} \quad +f_s u^* \quad +m^2\frac{s^{*2}}{r}\sin\phi \quad +\frac{\kappa h}{p}\frac{1}{r}\left(\frac{\partial p}{\partial \varphi}-\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r}\frac{\partial r}{\partial \varphi}\right) \quad = F_v$$

$$\frac{dw}{dt}-m^2\frac{s^{*2}}{r} \quad -m^2 f_c^* u^* \quad +\frac{\kappa h}{p}\frac{\partial p}{\partial \zeta}\frac{\partial \zeta}{\partial r} \quad +g = F_w$$

$$\frac{dh}{dt}-\frac{\kappa h}{p}\frac{dp}{dt}=F_h$$

Deep Atmos Dyn
in spherical mapping
& generalized coordinates

$$\frac{\partial \rho^*}{\partial t}+m^2\frac{\partial \rho^* \frac{u^*}{r}}{\partial \lambda}+m^2\frac{\partial \rho^* \frac{v^*}{r}}{\partial \varphi}+\frac{\partial \rho^* \dot{\zeta}}{\partial \zeta}=F_\rho^*$$

Staniforth and Wood (2003)
Juang (2014) NCEP Office Note

$$\frac{dq_i}{dt}=F_{q_i}$$

$$p=\rho\kappa h$$

where

$$\frac{d()}{dt}=\frac{\partial()}{\partial t}+\dot{\lambda}\frac{\partial()}{\partial \lambda}+\dot{\varphi}\frac{\partial()}{\partial \varphi}+\dot{\zeta}\frac{\partial()}{\partial \zeta}=\frac{\partial()}{\partial t}+m^2u^*\frac{\partial()}{r\partial \lambda}+m^2v^*\frac{\partial()}{r\partial \varphi}+\dot{\zeta}\frac{\partial()}{\partial \zeta} \quad ; \quad \rho^*=\rho\frac{r^2}{a^2}\frac{\partial r}{\partial \zeta}$$

$$f_s=2\Omega\sin\phi \;;\; f_c^*=2\Omega\cos^2\phi \;;\; g=g(r) \;;\; \kappa=\frac{R}{C_P} \;;\; \gamma=\frac{C_P}{C_V} \;;\; s^{*2}=u^{*2}+v^{*2}$$

Start from continuity equation to have similar to shallow and hydrostatic system to make as mass coordinates,
Mass at give area can be obtained by integral vertical as

$$Mass = \int_{\phi_1}^{\phi_2} \int_{\lambda_1}^{\lambda_2} \int_{\zeta_{sfc}}^{\zeta_{Top}} \rho r^2 \cos\phi \frac{\partial r}{\partial \zeta} d\zeta \, d\lambda \, d\phi$$

Then project this mass to earth surface

$$pressure = \frac{Mass \; \overline{g}}{Area} = \frac{\iiint \rho \overline{g} r^2 \cos\phi \frac{\partial r}{\partial \zeta} d\zeta \, d\lambda \, d\phi}{\iint a^2 \cos\phi \, d\lambda \, d\phi}$$

We define it as coordinate pressure

Previous integral can be deduced to be only in vertical as

$$\int_{\zeta_{surface}}^{\zeta_{TOP}} \rho \bar{g} \frac{r^2}{a^2} \frac{\partial r}{\partial \zeta} d\zeta = \tilde{\bar{p}}_{\zeta_{surface}} = -\int_{\zeta_{surface}}^{\zeta_{TOP}} \frac{\partial \tilde{\bar{p}}}{\partial \zeta} d\zeta$$

Thus, we have $$\frac{\partial \tilde{\bar{p}}}{\partial \zeta} = -\rho \bar{g} \frac{r^2}{a^2} \frac{\partial r}{\partial \zeta} = -\rho^* \bar{g}$$

Put into continuity equation

$$\frac{\partial \rho^*}{\partial t} + m^2 \frac{\partial \rho^* \frac{u^*}{r}}{\partial \lambda} + m^2 \frac{\partial \rho^* \frac{v^*}{r}}{\partial \varphi} + \frac{\partial \rho^* \dot{\zeta}}{\partial \zeta} = 0$$

We have $$\frac{\partial \frac{\partial \tilde{\bar{p}}}{\partial \zeta}}{\partial t} + m^2 \left(\frac{\partial \frac{\partial \tilde{\bar{p}}}{\partial \zeta} \frac{u^*}{r}}{\partial \lambda} + \frac{\partial \frac{\partial \tilde{\bar{p}}}{\partial \zeta} \frac{v^*}{r}}{\partial \varphi} \right) + \frac{\partial \frac{\partial \tilde{\bar{p}}}{\partial \zeta} \dot{\zeta}}{\partial \zeta} = 0$$

Since $\frac{\partial \tilde{\bar{p}}}{\partial \zeta} = -\rho^* g$ and $\rho^* > 0$

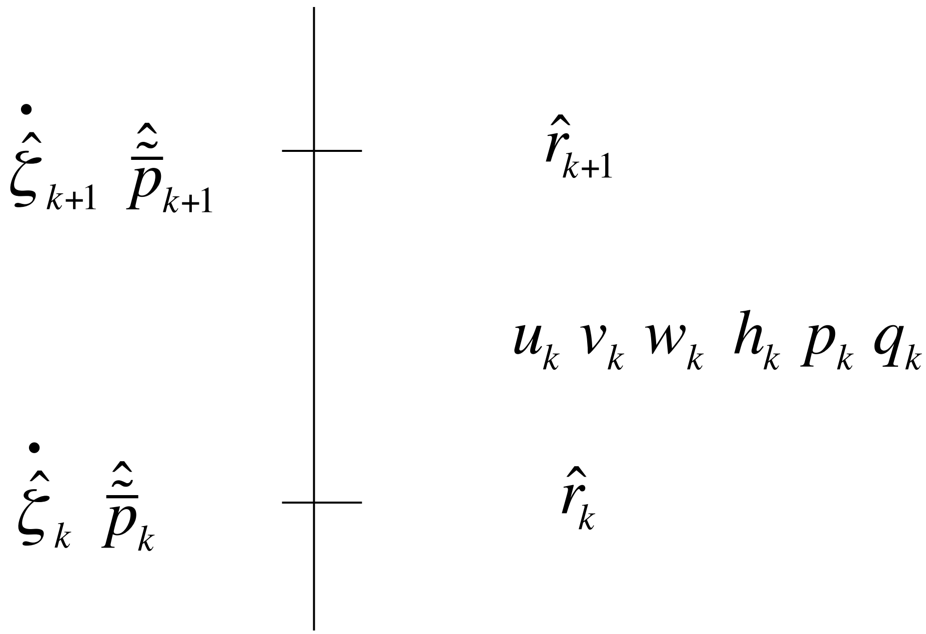
Thus $\tilde{\bar{p}}$ is monotone with vertical coordinate

We can use it for coordinate definition and we can call it height weighted coordinate pressure or simply coordinate pressure

So we can have $\hat{\tilde{\bar{p}}}_k = \hat{A}_k + \hat{B}_k \tilde{\bar{p}}_s + \hat{C}_k \left(\frac{h_{k-1} + h_k}{h_{0k-1} + h_{0k}} \right)^{C_{pd}/R_d}$

For opr compatibility, we use $\hat{\tilde{\bar{p}}}_k = \hat{A}_k + \hat{B}_k \tilde{\bar{p}}_s$

k=K , last layer at the top



k=1, the first layer next to ground

For dynamics, all source terms are set to be zero !

Vertical integral angular momentum principle, we have

$$\int_{\zeta_S}^{\zeta_T} \frac{\partial \tilde{\bar{p}}}{\partial \zeta} \frac{\kappa h}{p} \left(\frac{\partial p}{\partial \lambda} - \frac{\partial p}{\partial \zeta} \frac{\partial \zeta}{\partial r} \frac{\partial r}{\partial \lambda} \right) d\zeta = \int_{\zeta_S}^{\zeta_T} \left(\frac{\partial \tilde{\bar{p}}}{\partial \zeta} \frac{\kappa h}{p} \frac{\partial p}{\partial \lambda} - \frac{\partial \tilde{\bar{p}}}{\partial \zeta} \frac{\kappa h}{p} \frac{\partial p}{\partial \zeta} \frac{\partial \zeta}{\partial r} \frac{\partial r}{\partial \lambda} \right) d\zeta$$

$$= \int_{\zeta_S}^{\zeta_T} \left(-\frac{\partial r}{\partial \zeta} \frac{\bar{g} r^2}{a^2} \frac{\partial p}{\partial \lambda} + \frac{\bar{g} r^2}{a^2} \frac{\partial p}{\partial \zeta} \frac{\partial r}{\partial \lambda} \right) d\zeta$$

$$= \int_{\zeta_S}^{\zeta_T} \frac{\bar{g}}{3a^2} \left(-\frac{\partial r^3}{\partial \zeta} \frac{\partial p}{\partial \lambda} + \frac{\partial p}{\partial \zeta} \frac{\partial r^3}{\partial \lambda} \right) d\zeta$$

$$= \int_{\zeta_S}^{\zeta_T} \frac{\bar{g}}{3a^2} \left(-\frac{\partial r^3}{\partial \zeta} \frac{\partial p}{\partial \lambda} + \frac{\partial r^3 \frac{\partial p}{\partial \zeta}}{\partial \lambda} - r^3 \frac{\partial \frac{\partial p}{\partial \zeta}}{\partial \lambda} \right) d\zeta$$

$$= \int_{\zeta_S}^{\zeta_T} \frac{\bar{g}}{3a^2} \left(\frac{\partial r^3 \frac{\partial p}{\partial \zeta}}{\partial \lambda} - \frac{\partial r^3 \frac{\partial p}{\partial \lambda}}{\partial \zeta} \right) d\zeta$$

$$= \frac{\partial}{\partial \lambda} \int_{\zeta_S}^{\zeta_T} \frac{\bar{g}}{3a^2} r^3 \frac{\partial p}{\partial \zeta} d\zeta - \frac{\bar{g} r_T^3}{3a^2} \frac{\partial p_T}{\partial \lambda} + \frac{\bar{g} r_S^3}{3a^2} \frac{\partial p_S}{\partial \lambda}$$

Discretize the fourth and last LHSs

$$\int_{\zeta_S}^{\zeta_T} \frac{\bar{g}}{3a^2}\left(-\frac{\partial r^3}{\partial \zeta}\frac{\partial p}{\partial \lambda} + \frac{\partial r^3 \frac{\partial p}{\partial \zeta}}{\partial \lambda} - r^3 \frac{\partial \frac{\partial p}{\partial \zeta}}{\partial \lambda} \right) d\zeta = \frac{\partial}{\partial \lambda}\int_{\zeta_S}^{\zeta_T} \frac{\bar{g}}{3a^2} r^3 \frac{\partial p}{\partial \zeta} d\zeta - \frac{\bar{g} r_T^3}{3a^2}\frac{\partial p_T}{\partial \lambda} + \frac{\bar{g} r_S^3}{3a^2}\frac{\partial p_S}{\partial \lambda}$$

With P top zero, we have

$$\sum_{k=1}^{K}\left[\left(\hat{r}_{k+1}^3 - \hat{r}_k^3 \right)\frac{\partial p_k}{\partial \lambda} + r_k^3 \left(\frac{\partial \hat{p}_{k+1}}{\partial \lambda} - \frac{\partial \hat{p}_k}{\partial \lambda} \right) \right] = -\hat{r}_S^3 \frac{\partial p_S}{\partial \lambda}$$

let $p_k = f\left(\hat{p}_{k+1}, \hat{p}_k \right)$ so $p_k = \frac{\partial f_k}{\partial \hat{p}_{k+1}} \hat{p}_{k+1} + \frac{\partial f_k}{\partial \hat{p}_k} \hat{p}_k$ and $\frac{\partial f_k}{\partial \hat{p}_{k+1}} + \frac{\partial f_k}{\partial \hat{p}_k} = 1$

Then we have

$$\sum_{k=1}^{K}\left[\left\langle \left(\hat{r}_{k+1}^3 - \hat{r}_k^3 \right)\frac{\partial f_k}{\partial \hat{p}_{k+1}} + r_k^3 \right\rangle \frac{\partial \hat{p}_{k+1}}{\partial \lambda} + \left\langle \left(\hat{r}_{k+1}^3 - \hat{r}_k^3 \right)\frac{\partial f_k}{\partial \hat{p}_k} - r_k^3 \right\rangle \frac{\partial \hat{p}_k}{\partial \lambda} \right] = -\hat{r}_S^3 \frac{\partial p_S}{\partial \lambda}$$

For simplicity, we let

$$\frac{\partial f_k}{\partial \hat{p}_k} = \frac{\partial f_k}{\partial \hat{p}_{k+1}} = \frac{1}{2} \quad \text{so} \quad r_k^3 = \frac{1}{2}\hat{r}_{k+1}^3 + \frac{1}{2}\hat{r}_k^3$$

since $$\hat{r}_{k+1}^3 = \hat{r}_k^3 + 3\left(\frac{\kappa h}{p\bar{g}}\right)_k a^2 \left(\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}\right)$$

so $$r_k^3 = \hat{r}_k^3 + \frac{3}{2}\left(\frac{\kappa h}{p\bar{g}}\right)_k a^2 \left(\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}\right)$$

remove layer value we have $$\hat{r}_{k+1}^3 = \hat{r}_k^3 + 3\frac{a^2}{\bar{g}}\kappa_k h_k \frac{\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}}{p_k}$$

so $$r_k^3 = \hat{r}_1^3 + 3\frac{a^2}{\bar{g}}\sum_{i=1}^{k-1}\left[\kappa_i h_i \frac{\hat{\tilde{\bar{p}}}_i - \hat{\tilde{\bar{p}}}_{i+1}}{p_i}\right] + \frac{3}{2}\frac{a^2}{\bar{g}}\kappa_k h_k \frac{\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}}{p_k}$$

We have

$$\frac{\hat{r}_{k+1}^3}{a^2} = \frac{\hat{r}_k^3}{a^2} + 3\left(\frac{\kappa h}{p\overline{g}}\right)_k \left(\hat{\tilde{\overline{p}}}_k - \hat{\tilde{\overline{p}}}_{k+1}\right)$$

then

transform to spectral for r to do derivative, then transform back to grid space for nonlinear computing.

$$\frac{du_k^*}{dt} = -\frac{u_k^* w_k}{r_k} + f_s v_k^* - f_c^* w_k - \frac{\kappa_k h_k}{p_k}\frac{a}{r_k}\frac{\partial p_k}{a\partial\lambda} - \frac{\overline{g}}{6}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\tilde{\overline{p}}}_k - \hat{\tilde{\overline{p}}}_{k+1}}\frac{a}{r_k}\frac{\partial\left(\frac{\hat{r}_k^3}{a^2} + \frac{\hat{r}_{k+1}^3}{a^2}\right)}{a\partial\lambda}$$

$$\frac{dv_k^*}{dt} = -\frac{v_k^* w_k}{r_k} - f_s u_k^* - m^2\frac{s_k^{*2}}{r_k}\sin\phi - \frac{\kappa_k h_k}{p_k}\frac{a}{r_k}\frac{\partial p_k}{a\partial\varphi} - \frac{\overline{g}}{6}\frac{\hat{p}_k - \hat{p}_{k+1}}{\hat{\tilde{\overline{p}}}_k - \hat{\tilde{\overline{p}}}_{k+1}}\frac{a}{r_k}\frac{\partial\left(\frac{\hat{r}_k^3}{a^2} + \frac{\hat{r}_{k+1}^3}{a^2}\right)}{a\partial\varphi}$$

Select two forms of advection of PGF

$$\frac{1}{\gamma}\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{dp}{dt} = -p\nabla_H \bullet\left(\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right) + \frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial p}{\partial\zeta}\frac{\partial\zeta}{\partial r}\left(w - V_H\bullet\nabla_H r\right) - \frac{\partial}{\partial\zeta}\left(p\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\left(w - V_H\bullet\nabla_H r\right)\right)$$

$$= -p\nabla_H \bullet\left(\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right) - p\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\left(w - V_H\bullet\nabla_H r\right)\right)$$

Continue discretization, we have

$$\frac{dp}{dt} = -\gamma p\frac{1}{\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}}\nabla_H\bullet\left(\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}V_H\right) - \gamma p\frac{1}{\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}}\frac{\partial}{\partial\zeta}\left(\frac{\partial\tilde{\tilde{p}}}{\partial\zeta}\frac{\kappa h}{p}\frac{\partial\zeta}{\partial r}\left(w - V_H\bullet\nabla_H r\right)\right)$$

$$\left(\frac{dp}{dt}\right)_k = -\gamma_k p_k\left(\nabla_H\bullet\left(V_H\right) + \frac{1}{\Delta r^3}V_H\bullet\nabla_H\Delta r^3 - \frac{1}{\Delta r^3}\Delta\left(V_H\bullet\nabla_H r^3\right) + \frac{a^2}{r^2}\frac{\Delta\tilde{w}}{\Delta r}\right)_k$$

$$= -\gamma_k p_k\left(\left(\frac{\partial\dot{\lambda}}{\partial\lambda} + \frac{\partial\dot{\mu}}{\partial\mu}\right)_k - \frac{\frac{\dot{\lambda}_{k-1} - \dot{\lambda}_k}{2}\frac{\partial\hat{r}_k^3}{\partial\lambda} + \frac{\dot{\lambda}_k - \dot{\lambda}_{k+1}}{2}\frac{\partial\hat{r}_{k+1}^3}{\partial\lambda} + \frac{\dot{\mu}_{k-1} - \dot{\mu}_k}{2}\frac{\partial\hat{r}_k^3}{\partial\mu} + \frac{\dot{\mu}_k - \dot{\mu}_{k+1}}{2}\frac{\partial\hat{r}_{k+1}^3}{\partial\mu}}{\Delta r^3} + \left(\frac{a^2}{r^2}\frac{\Delta\tilde{w}}{\Delta r}\right)_k\right)$$

$$= -\gamma_k p_k\left(m^2\left[\left(\frac{\partial\frac{u^*}{r}}{\partial\lambda} + \frac{\partial\frac{v^*}{r}}{\partial\varphi}\right)_k - \frac{\left(\frac{u_{k-1}^*}{r_{k-1}} - \frac{u_k^*}{r_k}\right)\frac{\partial\hat{r}_k^3}{\partial\lambda} + \left(\frac{u_k^*}{r_k} - \frac{u_{k+1}^*}{r_{k+1}}\right)\frac{\partial\hat{r}_{k+1}^3}{\partial\lambda} + \left(\frac{v_{k-1}^*}{r_{k-1}} - \frac{v_k^*}{r_k}\right)\frac{\partial\hat{r}_k^3}{\partial\varphi} + \left(\frac{v_k^*}{r_k} - \frac{v_{k+1}^*}{r_{k+1}}\right)\frac{\partial\hat{r}_{k+1}^3}{\partial\varphi}}{2\Delta r^3}\right] + \left(\frac{a^2}{r^2}\frac{\Delta\tilde{w}}{\Delta r}\right)_k\right)$$

So we use dp/dt from momentum equation to thermodynamic equation, thus total energy will be conserved.

$$\frac{dh}{dt} - \frac{\kappa h}{p}\frac{dp}{dt} = 0$$

so

$$\frac{dh_k}{dt} = -\gamma_k \kappa_k h_k \left\{ m^2 \left(\frac{\partial \frac{u^*}{r}}{\partial \lambda} + \frac{\partial \frac{v^*}{r}}{\partial \varphi} \right)_k - m^2 \frac{\left(\frac{u^*_{k-1}}{r_{k-1}} - \frac{u^*_k}{r_k}\right)\frac{\partial \hat{r}^3_k}{\partial \lambda} + \left(\frac{u^*_k}{r_k} - \frac{u^*_{k+1}}{r_{k+1}}\right)\frac{\partial \hat{r}^3_{k+1}}{\partial \lambda} + \left(\frac{v^*_{k-1}}{r_{k-1}} - \frac{v^*_k}{r_k}\right)\frac{\partial \hat{r}^3_k}{\partial \varphi} + \left(\frac{v^*_k}{r_k} - \frac{v^*_{k+1}}{r_{k+1}}\right)\frac{\partial \hat{r}^3_{k+1}}{\partial \varphi}}{2\Delta r^3} + \left(\frac{a^2}{r^2}\frac{\Delta \tilde{w}}{\Delta r}\right)_k \right\}$$

where $\tilde{w} = \frac{r^2}{a^2} w$

$$\frac{dw}{dt} = m^2 \frac{s^{*2}}{r} + m^2 f_c^* u^* - \frac{\kappa h}{p} \frac{\partial p}{\partial \zeta} \frac{\partial \zeta}{\partial r} - g = m^2 \frac{s^{*2}}{r} + m^2 f_c^* u^* + \bar{g} \frac{r^2 \Delta p}{a^2 \Delta \tilde{\bar{p}}} - g$$

$$\frac{dr^2 w}{dt} = 2rw^2 + r^2 \frac{dw}{dt} = 2\frac{\left(r^2 w\right)^2}{r^3} + m^2 r s^{*2} + m^2 r^2 f_c^* u^* + \bar{g} a^2 \left(\frac{r^4 \Delta p}{a^4 \Delta \tilde{\bar{p}}} - 1 \right)$$

and $\tilde{w} = \frac{r^2}{a^2} w$

so

$$\frac{d\tilde{w}}{dt} = 2a^2 \frac{\tilde{w}^2}{r^3} + m^2 \frac{r}{a^2} s^{*2} + m^2 \frac{r^2}{a^2} f_c^* u^* + \bar{g} \left(\frac{r^4 \Delta p}{a^4 \Delta \tilde{\bar{p}}} - 1 \right)$$

Since $r_k^2 w_k = \frac{1}{2} \hat{r}_{k+1}^2 \hat{w}_{k+1} + \frac{1}{2} \hat{r}_k^2 \hat{w}_k$ so $\tilde{w}_k = \frac{1}{2} \hat{\tilde{w}}_{k+1} + \frac{1}{2} \hat{\tilde{w}}_k$

And BC $\hat{w}_1 = m^2 \frac{u_1^*}{\hat{r}_1} \frac{\partial \hat{r}_1}{\partial \lambda} + m^2 \frac{v_1^*}{\hat{r}_1} \frac{\partial \hat{r}_1}{\partial \varphi}$

so $\hat{\tilde{w}}_1 = m^2 \frac{\hat{r}_1}{a} \left(u_1^* \frac{\partial \hat{r}_1}{a \partial \lambda} + v_1^* \frac{\partial \hat{r}_1}{a \partial \varphi} \right)$ Then we have w at all levels

Linearize for SISL

- Collect discretized equations
- Linearization
- Matrixes for semi-implicit
- Add semi-Lagrangian with semi-implicit

To linearize, we define a base state

start

$$\hat{p}_{01} = 101.326 \quad ; \quad \hat{r}_{01} = a = 6371220 \quad ; \quad h_{ok} = C_{Pd} T_{0k} \quad ; \quad T_{0k} = 300$$

$$\hat{\tilde{\bar{p}}}_{0k} = \hat{A}_k + \hat{B}_k \hat{p}_{01} \quad ; \quad \Delta\tilde{\bar{p}}_{0k} = \hat{\tilde{\bar{p}}}_{0k} - \hat{\tilde{\bar{p}}}_{0k+1} \quad ; \quad k_0 = R_d / C_{Pd}$$

Since p equals to p bar, so $$\frac{\partial \tilde{\bar{p}}}{\partial \zeta} = -\rho g \frac{r^2}{a^2} \frac{\partial r}{\partial \zeta} = -\frac{\tilde{\bar{p}} g}{\kappa h} \frac{\partial r}{\partial \zeta}$$

So we can get all r at interface as $$\hat{r}_{0k+1} = \hat{r}_{0k} + \frac{\kappa_{0k} h_{0k}}{\bar{g}} \ln \frac{\hat{\tilde{\bar{p}}}_{0k}}{\hat{\tilde{\bar{p}}}_{0k+1}}$$

then use $$\hat{p}_{0k} - \hat{p}_{0k+1} = \left(\hat{\tilde{\bar{p}}}_{0k} - \hat{\tilde{\bar{p}}}_{0k+1} \right) \frac{a^2}{r_{0k}^2} \quad ; \quad p_{0k} = \frac{1}{2} \left(\hat{p}_{0k} + \hat{p}_{0k+1} \right)$$

then $$r_{0k}^3 = \frac{1}{2} \left(\hat{r}_{0k}^3 + \hat{r}_{0k+1}^3 \right) \qquad \Delta p_{0k} = \hat{p}_{0k} - \hat{p}_{0k+1}$$

and $$\varepsilon_{0k} = \frac{r_{0k}}{a}$$

Summary of linear equations in matrix/vector short form

$$\left(\frac{D\tilde{\bar{p}}_s}{Dt}\right)_L = -\Pi_{1i} D_i^*$$

$$\left(\frac{d\tilde{w}_k}{dt}\right)_L = -b_{0k}\tilde{\bar{p}}_s + \Gamma_{ki} p_i$$

$$\left(\frac{dp_k}{dt}\right)_L = -d_{0k} D_k^* + \mathrm{M}_{ki}\tilde{w}_i$$

$$\left(\frac{dh_k}{dt}\right)_L = -f_{0k} D_k^* + \mathrm{Z}_{ki}\tilde{w}_i$$

$$\left(\frac{dD_k^*}{dt}\right)_L = \frac{n(n+1)}{a^2}\left(\mathrm{A}_{ki} p_i + \mathrm{B}_{ki} h_i + e_k p_s\right)$$

$$\delta D_k^* - \frac{n(n+1)}{a^2}\alpha\delta t\left(\mathrm{A}_{ki}\delta p_i + \mathrm{B}_{ki}\delta h_i + e_k\delta\tilde{\bar{p}}_s\right) = S_{D*}$$

$$\delta\tilde{w}_k + \alpha\delta t\left(b_{0k}\delta\tilde{\bar{p}}_s - \Gamma_{ki}\delta p_i\right) = S_{\tilde{w}_k}$$

$$\delta h_k + \alpha\delta t\left(f_{0k}\delta D_k^* - \mathrm{Z}_{ki}\delta\tilde{w}_i\right) = S_{h_k}$$

$$\delta p_k + \alpha\delta t\left(d_{0k}\delta D_k^* - \mathrm{M}_{ki}\delta\tilde{w}_i\right) = S_{p_k}$$

$$\delta\tilde{\bar{p}}_s + \alpha\delta t\Pi_{1i}\delta D_i^* = S_{\tilde{\bar{p}}_s}$$

To solve it, we put ps and p into w to get w as function of D Then put w as function of D into h and p, thus we get p, h, and ps are all function of D, then put them into D equation to solve D

Hydrostatic IC

- Surface pressure
- Divergence/vorticity => U/V
- Use coordinate pressure to interpolation
- Use hydrostatic pressure as coordinate pressure and nonhydrostatic pressure
- Generate vertical velocity from w=dz/dt equation

From hydrostatic system, we have all p at interfaces as

$$\hat{p}_k = \hat{A}_k + \hat{B}_k p_s \qquad \& \qquad p_k = \frac{1}{2}\left(\hat{p}_k + \hat{p}_{k+1}\right)$$

From hydrostatic relation, we have

$$\hat{z}_{k+1} = \hat{z}_k + \left(\hat{p}_k - \hat{p}_{k+1}\right)\frac{\kappa_k h_k}{p_k g}$$

So coordinate pressure in hydrostatic system is

$$\left(\hat{\tilde{\bar{p}}}_k\right)_{hydro} = \left(\hat{\tilde{\bar{p}}}_{k+1}\right)_{hydro} + \frac{\left(a+\hat{z}_{k+1}\right)^3 - \left(a+\hat{z}_k\right)^3}{3a^2}\left(\frac{p\bar{g}}{\kappa h}\right)_k$$

The new nonhydrostatic coordinate pressure is

$$\boxed{\hat{\tilde{\bar{p}}}_k = \hat{A}_k + \hat{B}_k \hat{\tilde{\bar{p}}}_s}$$

So interpolate u, v, h, and q from hydro to new coordinates

Remaining p and w at deep atmospheric system

From coordinate pressure definition

$$\hat{r}_{k+1}^3 = \hat{r}_k^3 + 3\left(\frac{\kappa h}{p\bar{g}}\right)_k a^2 \left(\hat{\bar{\bar{p}}}_k - \hat{\bar{\bar{p}}}_{k+1}\right)$$

and hydrostatic relation

$$\hat{r}_{k+1} = \hat{r}_k + \left(\hat{p}_k - \hat{p}_{k+1}\right)\frac{\kappa_k h_k}{p_k \bar{g}}$$

combine both by eliminating kh/(pg), we have

$$\hat{p}_k - \hat{p}_{k+1} = 3\left(\frac{\hat{r}_{k+1} - \hat{r}_k}{\hat{r}_{k+1}^3 - \hat{r}_k^3}\right) a^2 \left(\hat{\bar{\bar{p}}}_k - \hat{\bar{\bar{p}}}_{k+1}\right)$$

can obtain p from top to surface

$$p_k = \frac{1}{2}\left(\hat{p}_k + \hat{p}_{k+1}\right)$$

Get vertical velocity w in deep atmospheric system

from $\hat{r}_{k+1}^3 = \hat{r}_k^3 + 3\left(\frac{\kappa h}{p\bar{g}}\right)_k a^2\left(\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}\right)$ Do d/dt

We have $$\hat{\tilde{w}}_{k+1} = \hat{\tilde{w}}_k + \frac{\kappa_k h_k}{p_k g}\Delta B_k \frac{d\ \hat{\tilde{\bar{p}}}_s}{dt} - \Delta\tilde{\bar{p}}_k \frac{\kappa_k h_k}{\gamma p_k^2 g}\frac{dp_k}{dt}$$

where

$$\frac{d\hat{\tilde{\bar{p}}}_s}{dt} = -\sum_{k=1}^{K} m^2\left(\hat{\tilde{\bar{p}}}_k - \hat{\tilde{\bar{p}}}_{k+1}\right)\left(\frac{\partial \frac{u^*}{r}}{\partial\lambda} + \frac{\partial \frac{v^*}{r}}{\partial\varphi}\right)_k$$

$$\left(\frac{dp}{dt}\right)_k = -\gamma_k p_k\left(\nabla_H \bullet \left(V_H\right) + \frac{1}{\Delta r^3}V_H \bullet \nabla_H \Delta r^3 - \frac{1}{\Delta r^3}\Delta\left(V_H \bullet \nabla_H r^3\right) + \frac{a^2}{r^2}\frac{\Delta\tilde{w}}{\Delta r}\right)_k$$

and $$\hat{\tilde{w}}_1 = m^2\frac{\hat{r}_1}{a}\left(u_1^*\frac{\partial\hat{r}_1}{a\partial\lambda} + v_1^*\frac{\partial\hat{r}_1}{a\partial\varphi}\right) \quad \& \quad \tilde{w}_k = \frac{1}{2}\hat{\tilde{w}}_{k+1} + \frac{1}{2}\hat{\tilde{w}}_k$$

Summary

- Deep atmospheric dynamics is discretized in generalized hybrid coordinates
- Semi-implicit semi-Lagrangian is used
- Initial condition with hydrostatic system is used
- It is suitable for very high resolution and coupling with space model
- Details can be found in Juang 2014, NCEP office note #477
- Coded and still under testing.