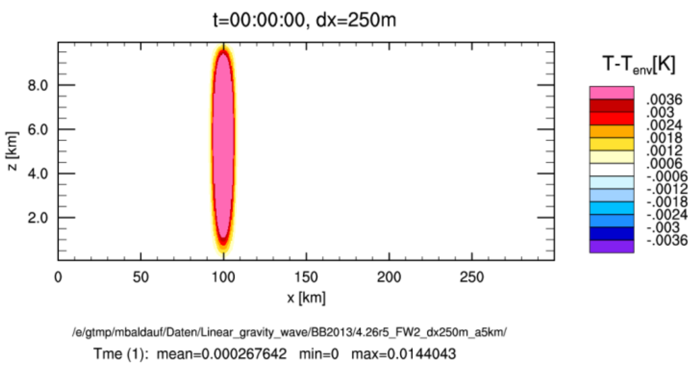
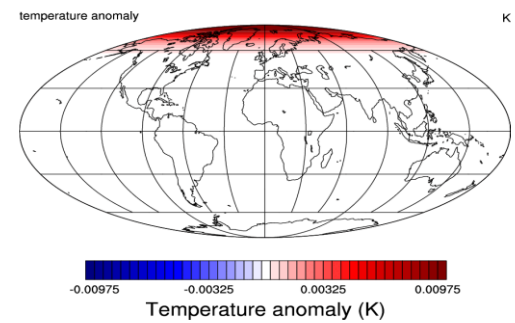
An analytical solution for gravity and sound wave expansion of the linearized compressible, non-hydrostatic Euler equations on the sphere

Michael Baldauf, Daniel Reinert, Günther Zängl



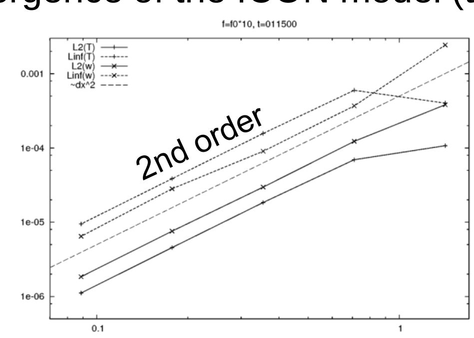


$$\hat{w}_{lm}(k_z,t) = -g\frac{\hat{\rho}_{lm}(k_z,t=0)}{\rho_s}\frac{1}{\beta^2-\alpha^2}\cdot\Bigg[-\alpha\sin\alpha t+\beta\sin\beta t + \left(f^2+c_s^2\frac{l(l+1)}{r_s^2}\right)\left(\frac{1}{\alpha}\sin\alpha t-\frac{1}{\beta}\sin\beta t\right)\Bigg]$$

Test scenarios

(A)Only gravity wave and sound wave expansion
(B)Additional Coriolis force (‚global f-plane approx. on a sphere‘)
(C)Additional advection by a solid body rotation velocity field $\boldsymbol{v}_0 = \boldsymbol{Q} \times \boldsymbol{r}$

Convergence of the ICON model (test (B)):



M. Baldauf. D. Reinert, G. Zängl (2014) QJRMS

PDEs on the sphere
Boulder / Colorado / USA
07-11 April 2014

Deutscher Wetterdienst
Wetter und Klima aus einer Hand

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Motivation

For the development of dynamical cores (or numerical methods in general) **idealized test cases** are an important evaluation tool.

- Idealized standard test cases with (at least approximated) analytic solutions:
  - stationary flow over mountains
    linear: *Queney (1947, ...), Smith (1979, ...) Adv Geophys, Baldauf (2008) COSMO-Newsl.*
    non-linear: *Long (1955) Tellus* for Boussinesq-approx. atmosphere
  - Balanced solutions on the sphere: *Staniforth, White (2011) ASL*
  - non-stationary, linear expansion of gravity waves in a channel
    Skamarock, Klemp (1994) MWR for Boussinesq-approx. atmosphere
- most of the other idealized tests only possess 'known solutions' gained from other numerical models.

There exist even fewer analytic solutions which use ***exactly the same equations*** as the numerical model used, i.e. in the sense that the numerical model ***converges to this solution***. One exception is given here:
linear expansion of gravity/sound waves on the sphere

Non-hydrostatic, compressible, shallow atmosphere, adiabatic, 3D Euler equations on a sphere:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p - g\mathbf{e}_z - 2\mathbf{\Omega}\times\mathbf{v}$$
$$\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho = -\rho\nabla\cdot\mathbf{v}$$
$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p = c_s^2\left(\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho\right)$$
$$c_s = \sqrt{\frac{c_p}{c_v}\frac{p}{\rho}}$$

Boundary conditions:
$w(r=r_s) = 0$
$w(r=r_s+H) = 0$ (rigid lid)

For an analytic solution only one further approximation is needed:
linearisation (= *controlled* approx.) around an **isothermal, steady, hydrostatic** atmosphere.

Solution strategy

Isothermal background state + shallow atmosphere approx.
→ Bretherton (1966) transformation
→ all coefficients of the linearized PDE system are constant

Shallow atmosphere approximation
- replace all prefactors $1/r$ → $1/r_s$
- in the divergence operator: omit the metric correction term ~ w/r
- apart from that all earth curvature metric terms are included
- (optional) Coriolis force by 'global f-plane approx. on a sphere':
  $2\boldsymbol{\Omega}(\lambda,\phi) = f\cdot \mathbf{e}_r(\lambda,\phi)$, f=const.
  (and $\mathbf{v}_0 = 0$)

Spectral representation of fields:

$$\psi(\lambda,\phi,r,t) = \sum_{k_z}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\hat{\psi}_{lm}(k_z,t)\cdot Y_{lm}(\phi,\lambda)\cdot e^{ik_z z}$$

Spherical harmonics:

$$Y_{lm}(\phi,\lambda) := N_{lm}\cdot P_{lm}(\sin\phi)\cdot e^{im\lambda} \qquad z = r - r_s$$

Time integration by Laplace transform

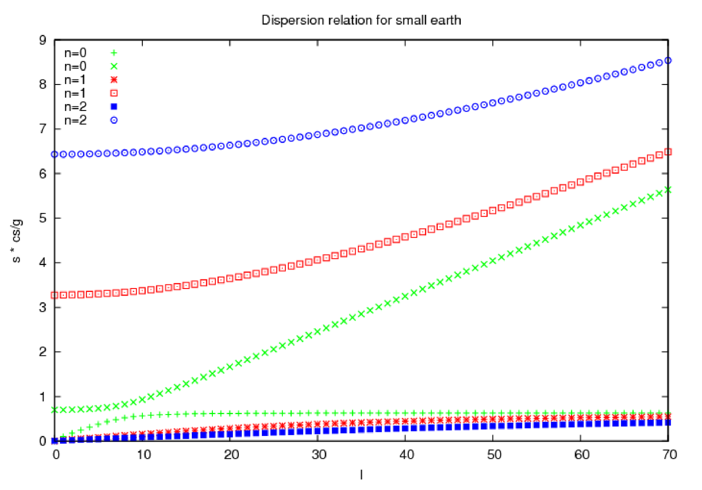
Analytic solution

for the vertical velocity w (Fourier component with k_z, spherical harmonic with l,m)

$$\hat{w}_{lm}(k_z,t) = -g\frac{\hat{\rho}_{lm}(k_z,t=0)}{\rho_s}\frac{1}{\beta^2-\alpha^2}\cdot\left[-\alpha\sin\alpha t + \beta\sin\beta t + \left(f^2 + c_s^2\frac{l(l+1)}{r_s^2}\right)\left(\frac{1}{\alpha}\sin\alpha t - \frac{1}{\beta}\sin\beta t\right)\right]$$

analogous expressions for $\hat{u}_{lm}(k_z, t)$, ...

The frequencies α, β are the gravity wave and acoustic branch, respectively, of the dispersion relation for compressible waves in a spherical channel of height H; $k_z = (\pi / H)\cdot n$:



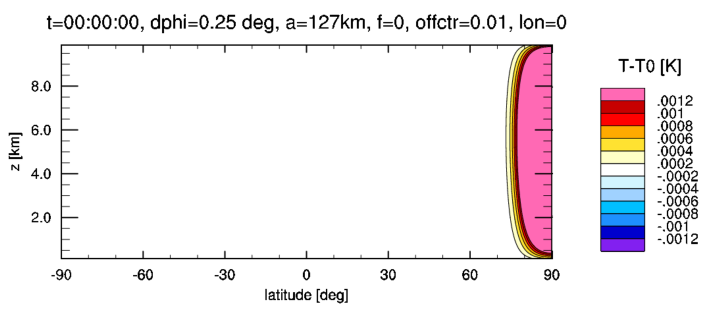
Test scenarios

(A) Only gravity wave and sound wave expansion

(B) Additional Coriolis force ('global f-plane approx. on a sphere')
→ test proper discretization of inertia-gravity modes, e.g. in a C-grid discretization.

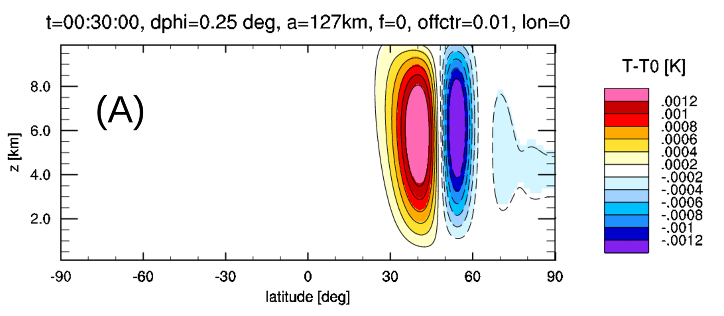
(C) Additional advection by a solid body rotation velocity field $\mathbf{v}_0 = \mathbf{Q}\times\mathbf{r}$
→ test the coupling of fast (buoyancy, sound) and slow (advection) processes.
Problem: solid body rotation field generates centrifugal forces.
Solution: choose $\mathbf{Q} = -\boldsymbol{\Omega}$
→ conforms to case (A) in the absolute system.

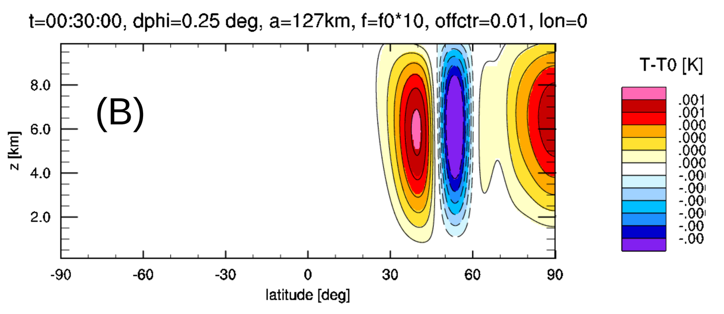
Test initialization



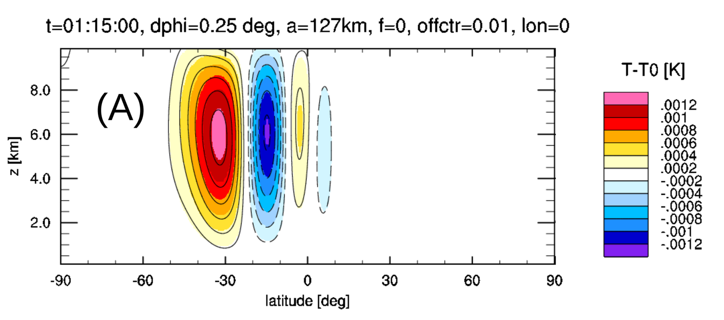
Time evolution of T'

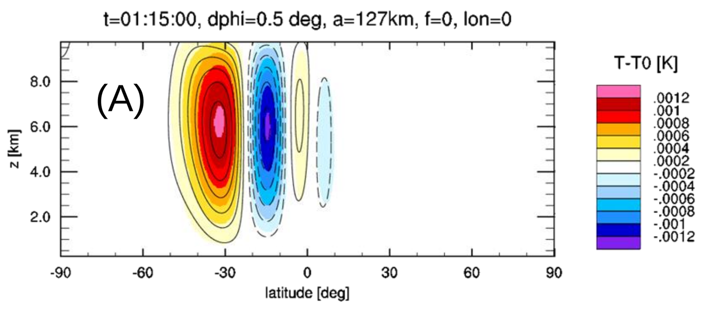
after 30 min.:

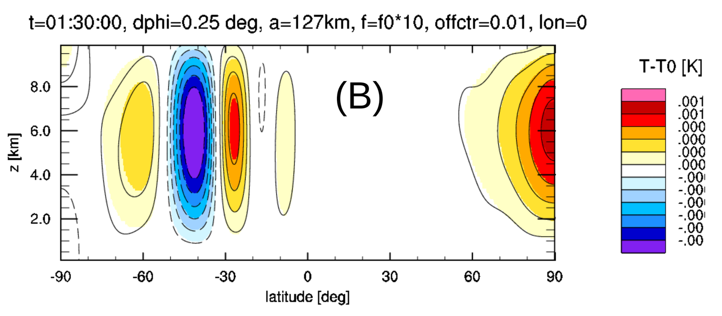




after 75 min.:







Black lines: analytic solution
Colours: ICON simulation

Small earth simulations

Wedi, Smolarkiewicz (2009) QJRMS

- $r_s = r_{earth} / 50$ ~ 127 km
  simulations with $\Delta\varphi$ ~ 1°... 0.0625°
  → Δx ~ 2.2 km ... 0.14 km
  → non-hydrostatic regime
- for runs *with* Coriolis force:
  $f = f_{earth}\cdot 10$ ~ 10^{-3} 1/s

→ dimensionless numbers
$Ro = 0.2\cdot Ro_{earth}$
$f / N = 10\cdot f_{earth} / N$ ~ 0.07

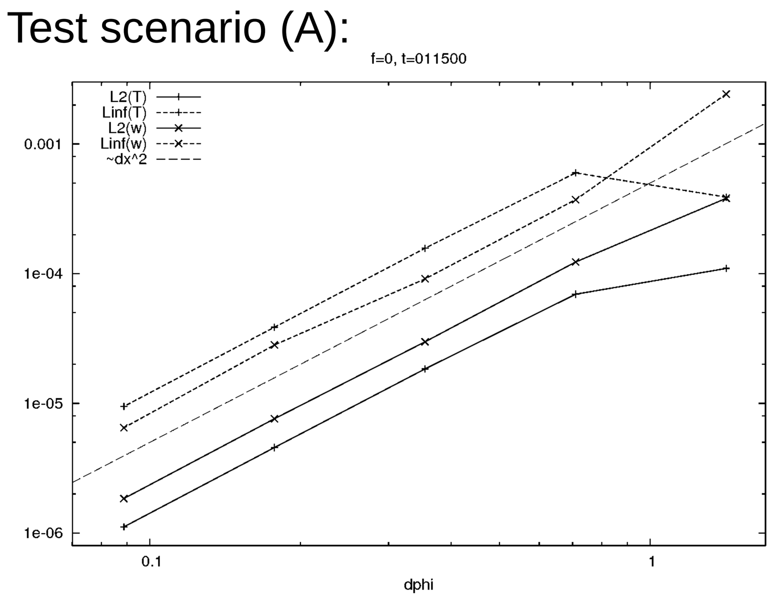
The ICON model

- joint development of DWD and MPI-M Hamburg
- **ICO**sahedral-triangular Arakawa-C grid
- **N**on-hydrostatic, compressible Eq. for v_n, w, ρ, $\rho\theta_v$ (or π)
- mixed FV- and FD- disretisations
- mass- and tracer-mass conservation
- predictor-corrector scheme (HE-VI)
- relevant for these tests:
  switch off any off-centring in the vertical implicit sound wave propagation → 2nd order convergence

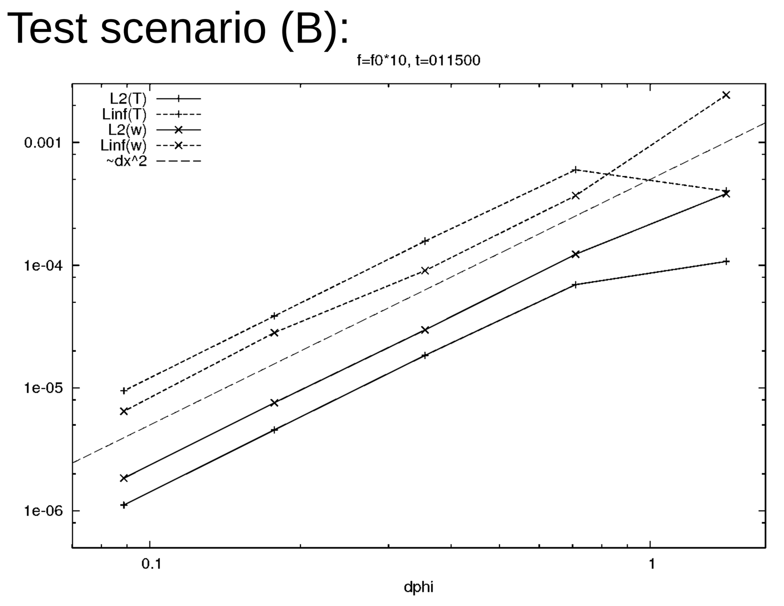
Convergence tests with ICON

Error measures L_2 (solid lines) and L_∞ (dashed) for T' (+ signs) and w (×) after 75 min.

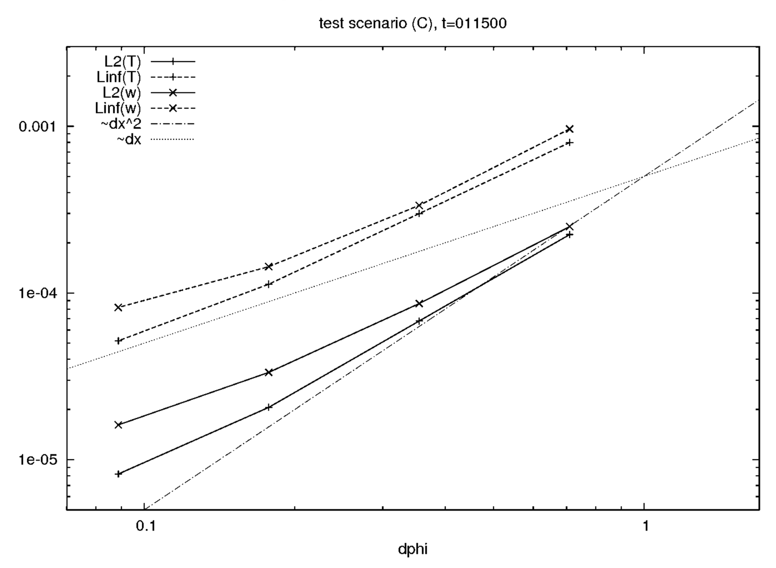
Test scenario (A):



Test scenario (B):



Test scenario (C):



References:

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