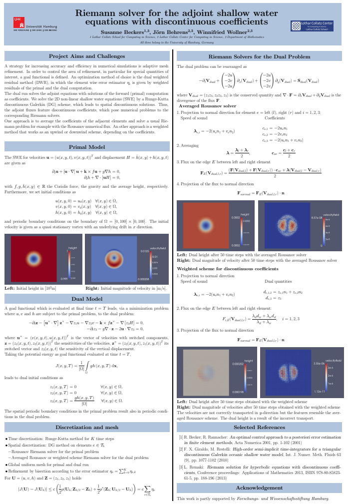
- Adaptive mesh refinement + quantities of interest computed by goal functional
  $\Rightarrow$ Dual weighted residual method
- $\eta_e =$ weighted residual of primal problem + weighted residual of dual problem
- Primal problem: 2D nonlinear Shallow water equations (SWE)
  Dual problem: adjoint equations of SWE depending on solutions of primal problem
- Discretization: Runge-Kutta discontinuous Galerkin scheme
  $\Rightarrow$ spatial discontinuous primal solutions
  $\Rightarrow$ discontinuous coefficients in dual problem
- Riemann problem with discontinuous coefficients
  1. Averaged coefficients + usual Riemann solver
  2. Weighted method (upwind/downwind)



Riemann solver for the adjoint shallow water equations with discontinuous coefficients

Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Susanne Beckers[1,3], Jörn Behrens[2,3], Winnifried Wollner[2,3]

1 Lothar Collatz School for Computing in Science, 2 Lothar Collatz Center for Computing in Science, 3 Department of Mathematics

All three belong to the University of Hamburg, Germany

Lothar Collatz Center for Computing in Science - Lothar Collatz School -

Project Aims and Challenges

A strategy for increasing accuracy and efficiency in numerical simulations is adaptive mesh refinement. In order to control the area of refinement, in particular for special quantities of interest, a goal functional is defined. An optimization method of choice is the dual weighted residual method (DWR), in which the element wise error estimator η_e is given by weighted residuals of the primal and the dual computation.
The dual run solves the adjoint equations with solutions of the forward (primal) computation as coefficients. We solve the 2D non-linear shallow water equations (SWE) by a Runge-Kutta discontinuous Galerkin (DG) scheme, which leads to spatial discontinuous solutions. Thus, the adjoint fluxes feature discontinuous coefficients, which pose numerical problems to the corresponding Riemann solvers.
One approach is to average the coefficients of the adjacent elements and solve a usual Riemann problem for example with the Rousanov numerical flux. An other approach is a weighted method that works as an upwind or downwind scheme, depending on the coefficients.

Primal Model

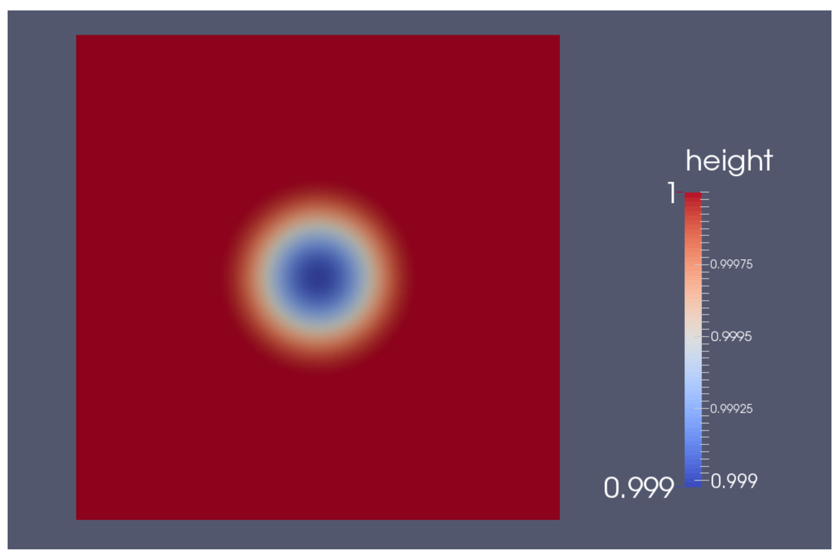
The SWE for velocities $\mathbf{u} = (u(x,y,t), v(x,y,t))^T$ and displacement $H = \bar{h}(x,y) + h(x,y,t)$ are given as

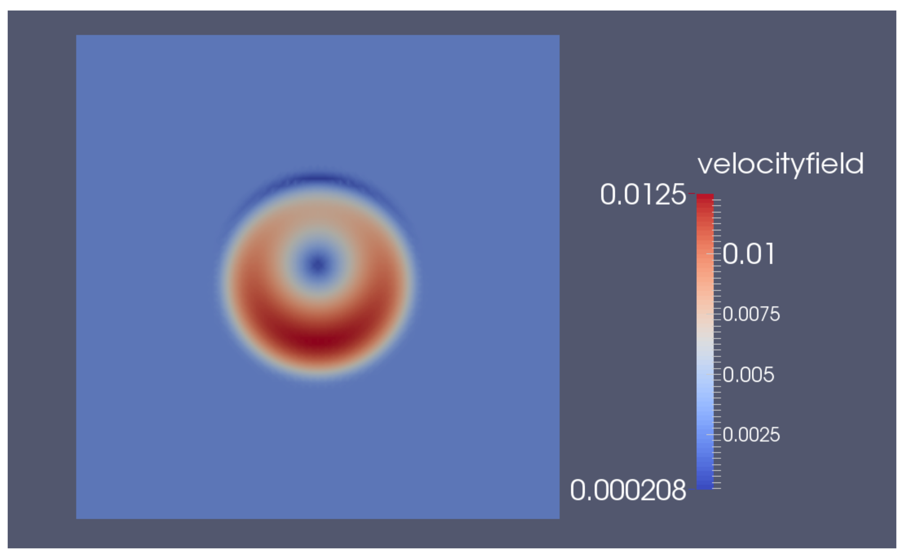
$$\begin{aligned} \partial_t \mathbf{u} + [\mathbf{u} \cdot \nabla]\, \mathbf{u} + \mathbf{k} \times f\mathbf{u} + g\nabla h &= 0, \\ \partial_t h + \nabla \cdot [\mathbf{u}H] &= 0, \end{aligned}$$

with $f, g, \bar{h}(x,y) \in \mathbb{R}$ the Coriolis force, the gravity and the average height, respectively. Furthermore, we set initial conditions as

$$\begin{aligned} u(x,y,0) &= u_0(x,y) \quad \forall (x,y) \in \Omega, \\ v(x,y,0) &= v_0(x,y) \quad \forall (x,y) \in \Omega, \\ h(x,y,0) &= h_0(x,y) \quad \forall (x,y) \in \Omega, \end{aligned}$$

and periodic boundary conditions on the boundary of $\Omega = [0,100] \times [0,100]$. The initial velocity is given as a quasi stationary vortex with an underlying drift in x direction.





Left: Initial height in $[10^3\mathrm{m}]$ **Right:** Initial magnitude of velocity in [m/s].

Dual Model

A goal functional which is evaluated at final time $t = T$ leads, via a minimization problem where u, v and h are subject to the primal problem, to the dual problem:

$$\begin{aligned} -\partial_t \mathbf{z} - \left[\mathbf{u}^\times \cdot \nabla\right] \mathbf{z}^\times - \nabla z_1 u - \nabla z_2 v - \mathbf{k} \times f\mathbf{z}^\times - \nabla\left[z_3 H\right] &= 0, \\ -\partial_t z_3 - g\nabla \cdot \mathbf{z} - 2\mathbf{u} \cdot \nabla z_3 &= 0, \end{aligned}$$

where $\mathbf{u}^\times = (v(x,y,t), u(x,y,t))^T$ is the vector of velocities with switched components, $\mathbf{z} = (z_1(x,y,t), z_2(x,y,t))^T$ the sensitivities of the velocities, $\mathbf{z}^\times = (z_2(x,y,t), z_1(x,y,t))^T$ its switched vector and $z_3(x,y,t)$ the sensitivity of the vertical displacement.
Taking the potential energy as goal functional evaluated at time $t = T$,

$$J(x,y,T) = \frac{1}{|\Omega|} \int_\Omega gh\,(x,y,T)\; \mathrm{d}\mathbf{x},$$

leads to dual initial conditions as

$$\begin{aligned} z_1(x,y,T) &= 0 && \forall (x,y) \in \Omega, \\ z_2(x,y,T) &= 0 && \forall (x,y) \in \Omega, \\ z_3(x,y,T) &= \frac{gh(x,y,T)}{|\Omega|} && \forall (x,y) \in \Omega. \end{aligned}$$

The spatial periodic boundary conditions in the primal problem result also in periodic conditions in the dual problem.

Discretization and mesh

- Time discretization: Runge-Kutta method for K time steps
- Spatial discretization: DG method on elements $e \in \mathcal{T}_h$
  - Rousanov Riemann solver for the primal problem
  - Averaged Rousanov or weighted scheme Riemann solver for the dual problem
- Global uniform mesh for primal and dual run
- Refinement by bisection according to the error estimator $\eta_e = \sum_{k=0}^{K} \eta_{e,k}$

For $\mathbf{U} = (u,v,h)$ and $\mathbf{Z} = (z_1, z_2, z_3)$ holds

$$|J(\mathbf{U}) - J(\mathbf{U}_h)| \le c\left(\frac{1}{2}\rho(\mathbf{U}_h; \mathbf{Z}_{h/2} - \mathbf{Z}_h) + \frac{1}{2}\rho^*(\mathbf{Z}_h; \mathbf{U}_{h/2} - \mathbf{U}_h)\right) = c \sum_{e \in \mathcal{T}_h} \eta_e$$

Riemann Solvers for the Dual Problem

The dual problem can be rearranged as

$$-\partial_t \mathbf{V}_{dual} + \begin{pmatrix} -2u \\ -2v \\ -2u \end{pmatrix} \partial_x(\mathbf{V}_{dual}) + \begin{pmatrix} -2u \\ -2v \\ -2v \end{pmatrix} \partial_y(\mathbf{V}_{dual}) = \mathbf{S}_{dual}(\mathbf{V}_{dual})$$

where $\mathbf{V}_{dual} = (z_1 z_3, z_2 z_3, z_3)$ is the conserved quantity and $\nabla \cdot \mathbf{F} = \partial_x \mathbf{V}_{dual} + \partial_y \mathbf{V}_{dual}$ is the divergence of the flux $\mathbf{F}$.

Averaged Rousanov solver

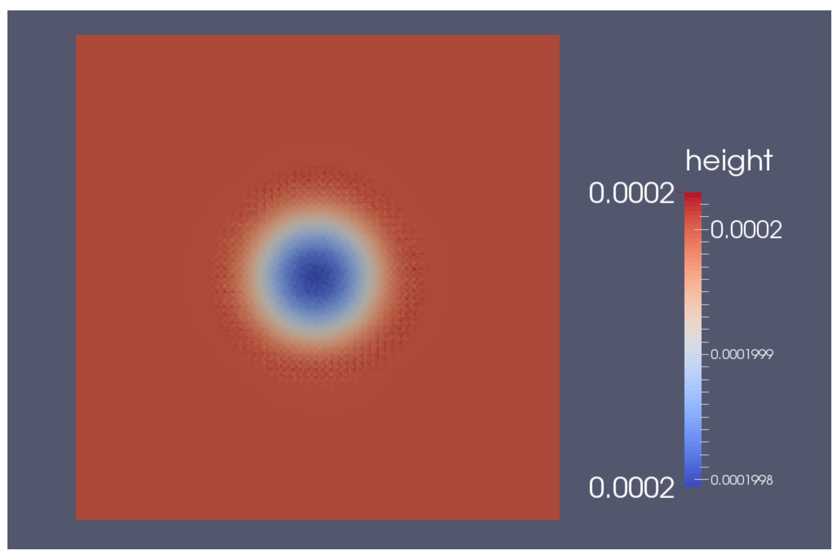
1. Projection to normal direction for element e = left (l), right (r) and $i = 1,2,3$:

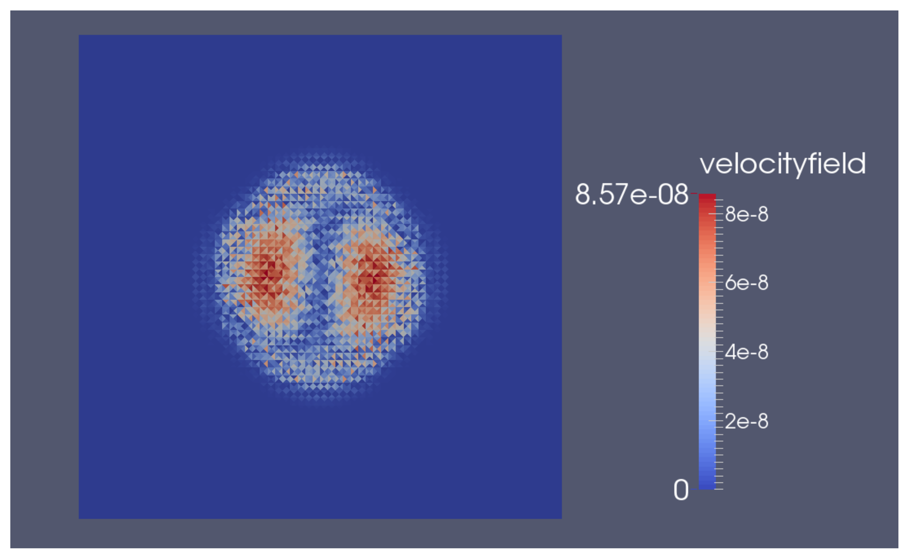
   Speed of sound
   $$\boldsymbol{\lambda}_{e,i} = -2(u_e n_1 + v_e n_2)$$
   Coefficients
   $$\begin{aligned} c_{e,1} &= -2u_e n_1 \\ c_{e,2} &= -2v_e n_2 \\ c_{e,3} &= -2(u_e n_1 + v_e n_2) \end{aligned}$$

2. Averaging
   $$\boldsymbol{\lambda} = \frac{\boldsymbol{\lambda}_l + \boldsymbol{\lambda}_r}{2}, \qquad \mathbf{c}_{av} = \frac{\mathbf{c}_l + \mathbf{c}_r}{2}$$

3. Flux on the edge E between left and right element
   $$\mathbf{F}_E(\mathbf{V}_{dual,l,r}) = \frac{(\mathbf{F}(\mathbf{V}_{dual,l}) + \mathbf{F}(\mathbf{V}_{dual,r})) \cdot \mathbf{c}_{av} + \boldsymbol{\lambda}(\mathbf{V}_{dual,l} - \mathbf{V}_{dual,r})}{2}$$

4. Projection of the flux to normal direction
   $$\mathbf{F}_{normal} = \mathbf{F}_E(\mathbf{V}_{dual,l,r}) \cdot \mathbf{n}$$





Left: Dual height after 50 time steps with the averaged Rousanov solver
Right: Dual magnitude of velocity after 50 time steps with the averaged Rousanov solver

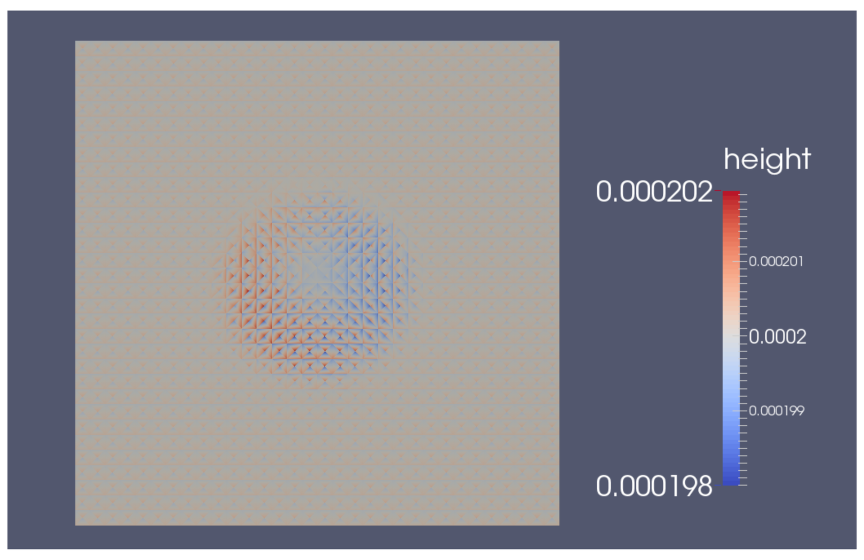
Weighted scheme for discontinuous coefficients

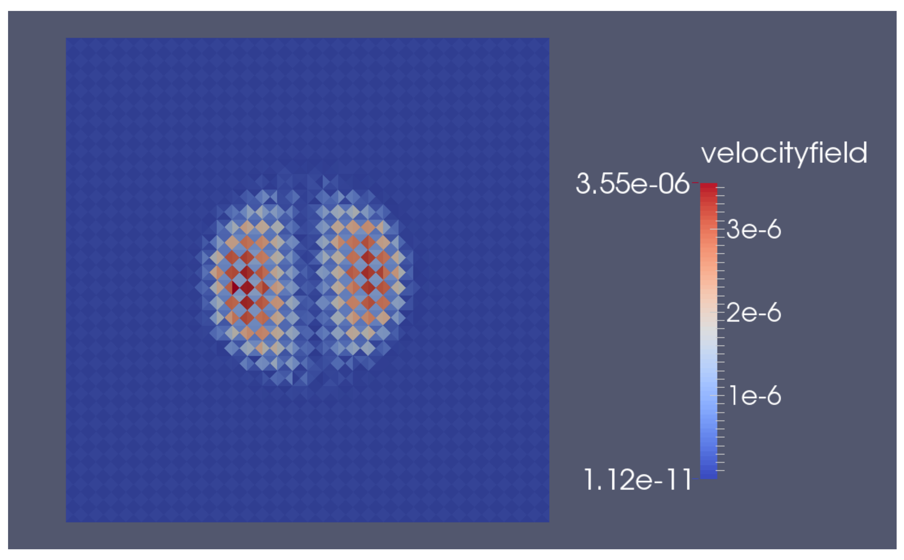
1. Projection to normal direction

   Speed of sound
   $$\boldsymbol{\lambda}_{e,i} = -2(u_e n_1 + v_e n_2)$$
   Dual quantities
   $$\begin{aligned} d_{e,1,2} &= z_{e,1} n_1 + z_{e,2} n_2 \\ d_{e,3} &= z_3 \end{aligned}$$

2. Flux on the edge E between left and right element:
   $$F_{i,E}(\mathbf{V}_{dual,l,r}) = \frac{\lambda_{il} d_{ir} + \lambda_{ir} d_{il}}{\lambda_{il} + \lambda_{ir}}, \quad i = 1,2,3$$

3. Projection of the flux to normal direction
   $$\mathbf{F}_{normal} = \mathbf{F}_E(\mathbf{V}_{dual,l,r}) \cdot \mathbf{n}$$





Left: Dual height after 50 time steps obtained with the weighted scheme
Right: Dual magnitude of velocities after 50 time steps obtained with the weighted scheme
The velocities are not correctly transported in y-direction but the features resemble the averaged Rousanov scheme. The dual height is a result of the incorrect transport.

Selected References

[1] R. Becker, R. Rannacher: *An optimal control approach to a posteriori error estimation in finite element methods*, Acta Numerica 2001, pp. 1-102 (2001)

[2] F. X. Giraldo, M. Restelli: *High-order semi-implicit time-integrators for a triangular discontinuous Galerkin oceanic shallow water model*, Int. J. Numer. Meth. Fluids 63 (9), pp. 1077-1102 (2010)

[3] L. Remaki: *Riemann solution for hyperbolic equations with discontinuous coefficients*, Conference proceedings: Applications of Mathematics 2013, ISBN 978-80-85823-61-5, pp. 188-196 (2013)

Acknowledgement

This work is partly supported by *Forschungs- und Wissenschaftsstiftung Hamburg*.