Unstable periodic orbits and normal modes

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Barotropic atmospheric system

The model is based on the barotropic vorticity equation on rotating sphere (=2D Navier-Stokes system +forcing +rotation +boundary&turbulent friction +orography).

$$\frac{\partial \Delta \psi}{\partial t} + J(\psi, \Delta\psi + l + H) = -\alpha \Delta \psi + \mu \Delta^2 \psi + f_{ext}.$$

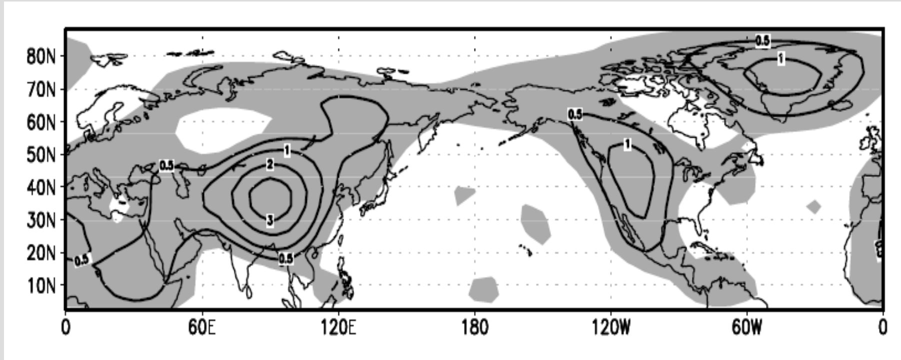
Δ **- Laplacian,** J **- Jacobian,** l **- Coriolis parameter,** ψ **- streamfunction,**

H **- orography,** f_{ext} **- external (constant in time) forcing, turbulent viscosity** μ

boundary layer friction α **, orography normalization** k

Galerkin method, T21 resolution (spherical harmonics m<22), Phase space dim=231

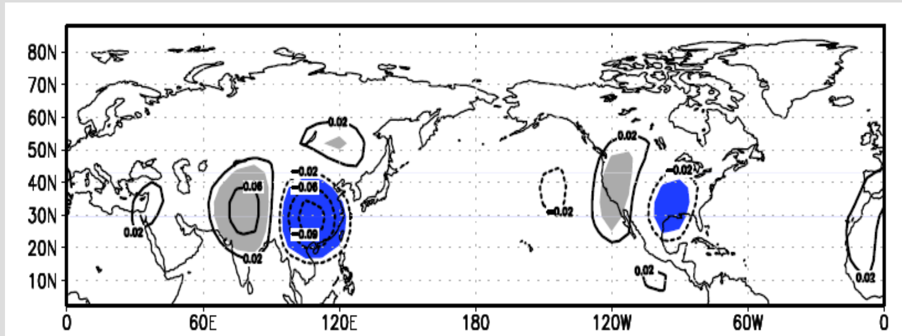
Model orography *H*



External forcing *f*

$$f_{ext=} J(\psi_r, \Delta\psi_r + l + H) + \alpha\Delta\psi_r - \mu\Delta^2\psi_r.$$

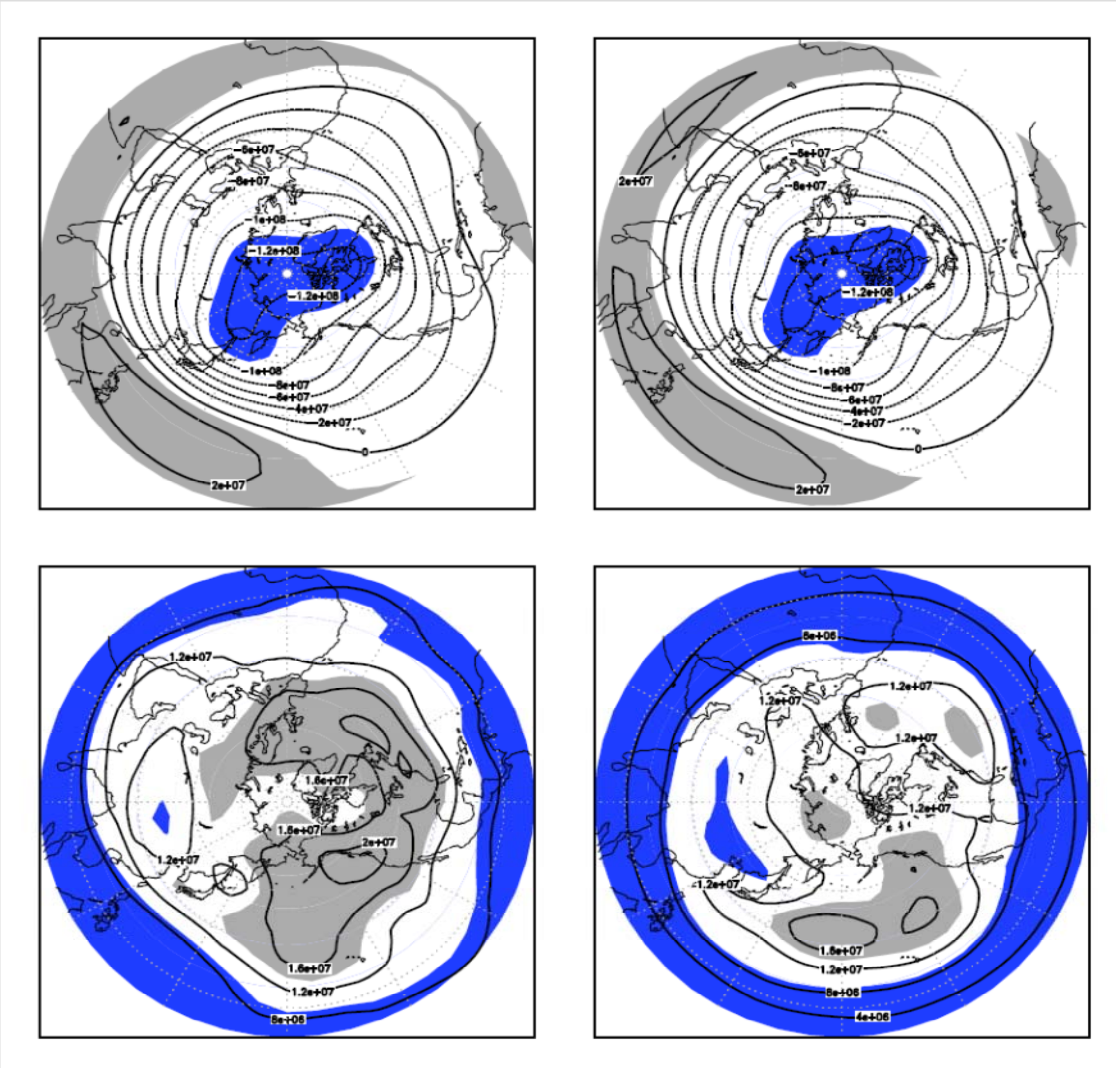
$\psi_r(t)$ is streamfunction on 300mb surface from 1960-1990 NCEP/NCAR reanalysis dataset



Parameters

turbulent viscosity	$\mu = 6 \cdot 10^{-5}$
boundary layer friction	$\alpha = 2 \cdot 10^{-3}$ i.e. 1/25days
orography normalization	$k = 0.14$

Model climates vs Nature



300mb NCEP data T21

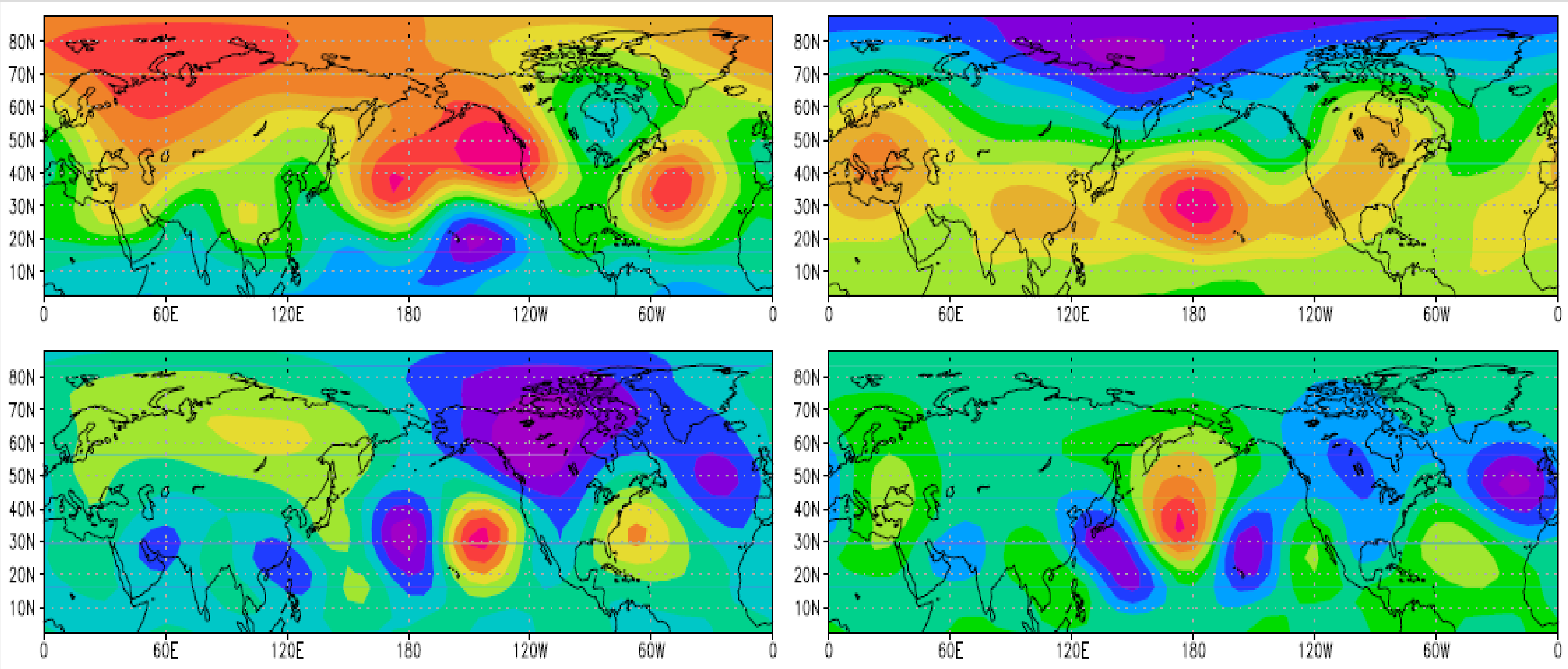
Average state

$$\overline{\psi} = \sum_k \psi(t_k)/K$$

Variance

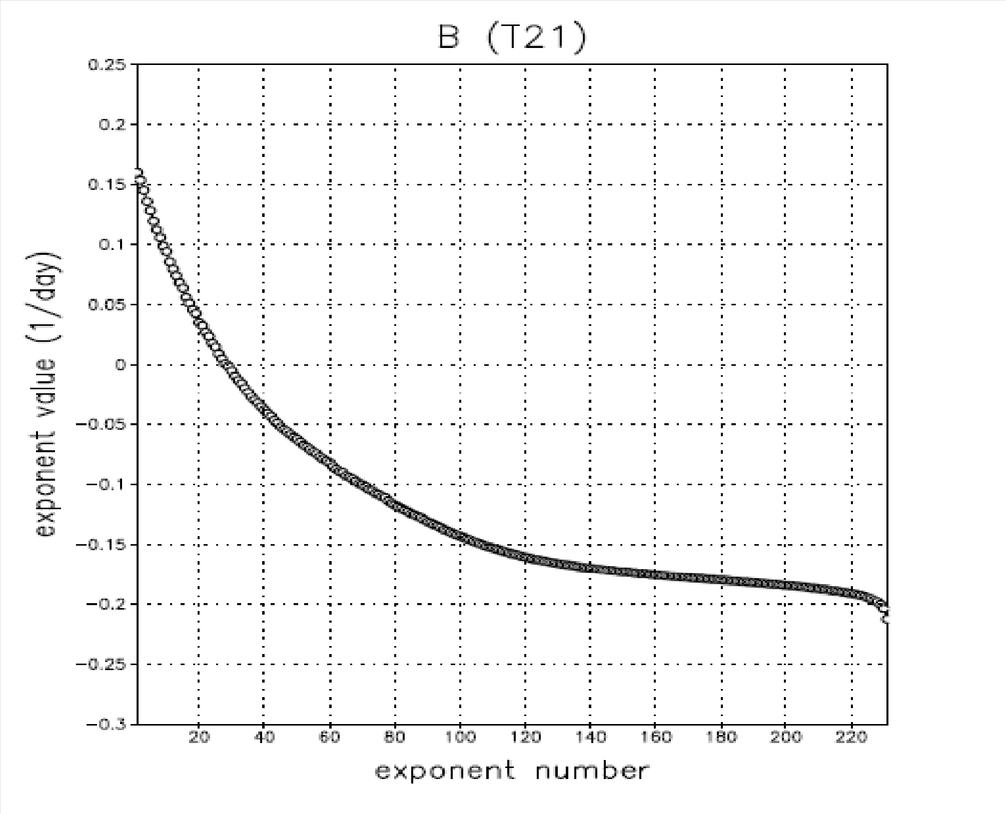
$$\sigma = (\frac{1}{K}\sum_k (\psi(t_k) - \overline{\psi}))^2)^{1/2}$$

Model modes of variability

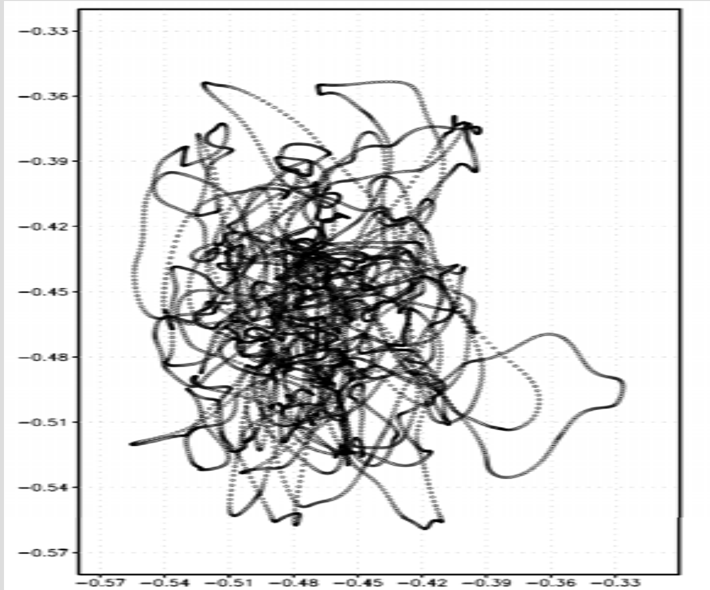


Streamfunction
EOF1,EOF2
EOF3,EOF4

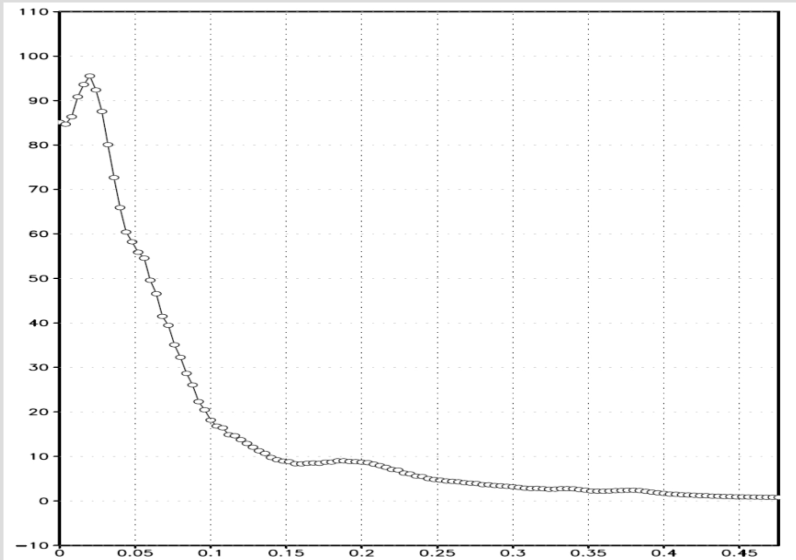
Attractor dimension equals to 62
28 positive Lyapunov exponents



Lyapunov spectrum for T21 models (Lyupunov time is 6 days)



Trajectory and PDF of the model (EOF1,EOF2) plane



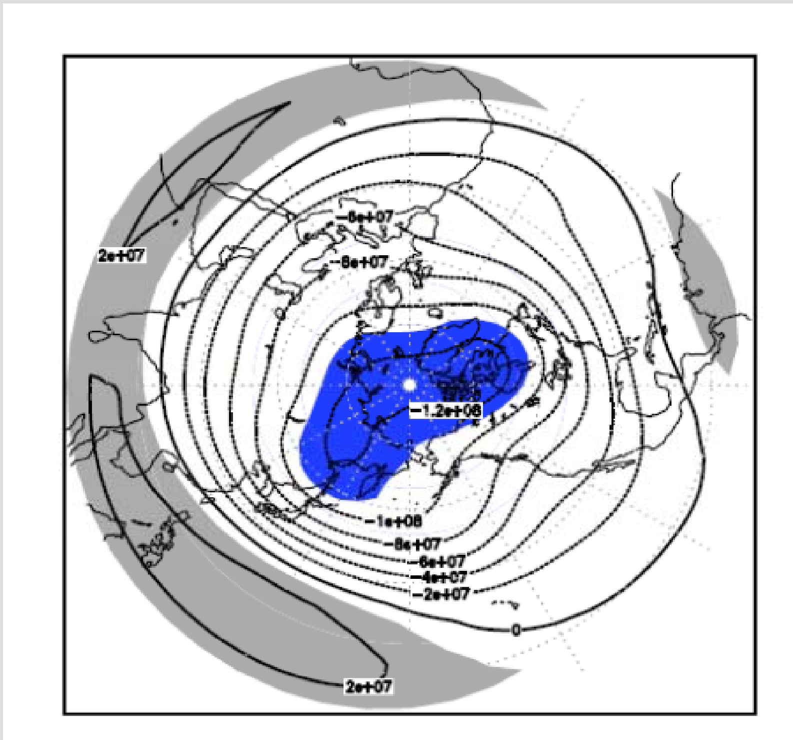
Power spectrum (kinetic energy)

Linearization

$$\frac{\partial \Delta h}{\partial t} + J(h, \Delta\overline{\psi} + l + H) + J(\overline{\psi}, \Delta h) + \alpha\Delta h - \mu\Delta^2 h = 0$$

$$\frac{\partial h}{\partial t} = Ah$$

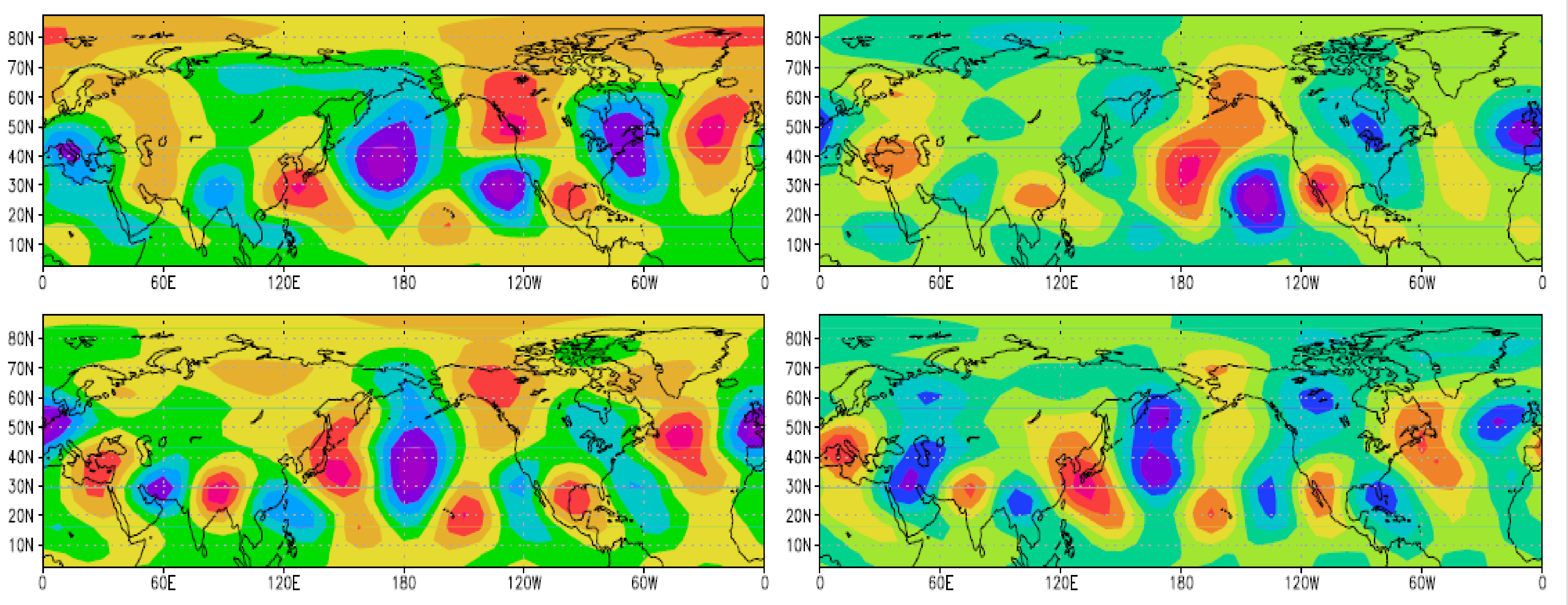
$$\overline{\psi}$$



Linear modes (streamfunction)

12 unstable modes (6 complex pairs), max(Re Λ) + 1/14days

Periods: 14, 8, 5, 56, 11, 31 days



EigVReal1, EigVCmplx1
EigVReal2, EigVCmplx2

Wavelike propagating structures with specific time-space scales

Standard EOFs → Bandpassed complex (rotational) EOFs

Complex (rotational) EOFs

Consider system trajectory $X_j(t^i)$ **(mean removed).**

Conventional EOFs are e.v. of $C = \langle XX^T \rangle$

1. **Apply direct (time) Fourier transform to every field component.** $X_j \rightarrow F[X_j] = a_k + ib_k$

2. **Apply π/2 time shift** $a_k + ib_k \rightarrow b_k - ia_k$

3. **Apply inverse Fourier transform** $b_k - ia_k \rightarrow F^{-1}(b_k - ia_k) = \tilde{X}$

4. **Make a complex time series and calculate its covariance matrix** $Z_j = X_j + i\tilde{X}_j, C_H = \langle ZZ^* \rangle$

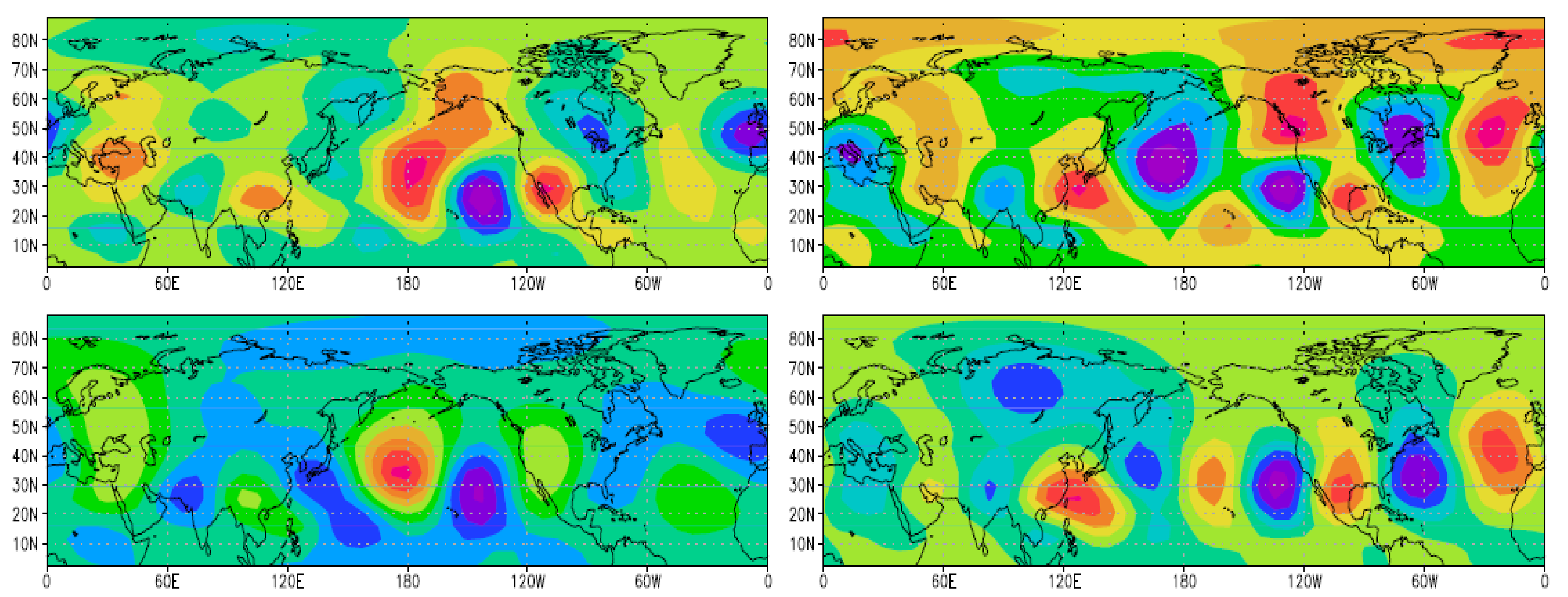
5. **Complex e.v. of** $C_H = \langle ZZ^* \rangle$ **are Hilbert (complex, rotational) EOFs**

Leading Hilbert EOF of the system defines 2D oscillating structures with largest amplitude

E.V.1 (cmplx, real) (top)

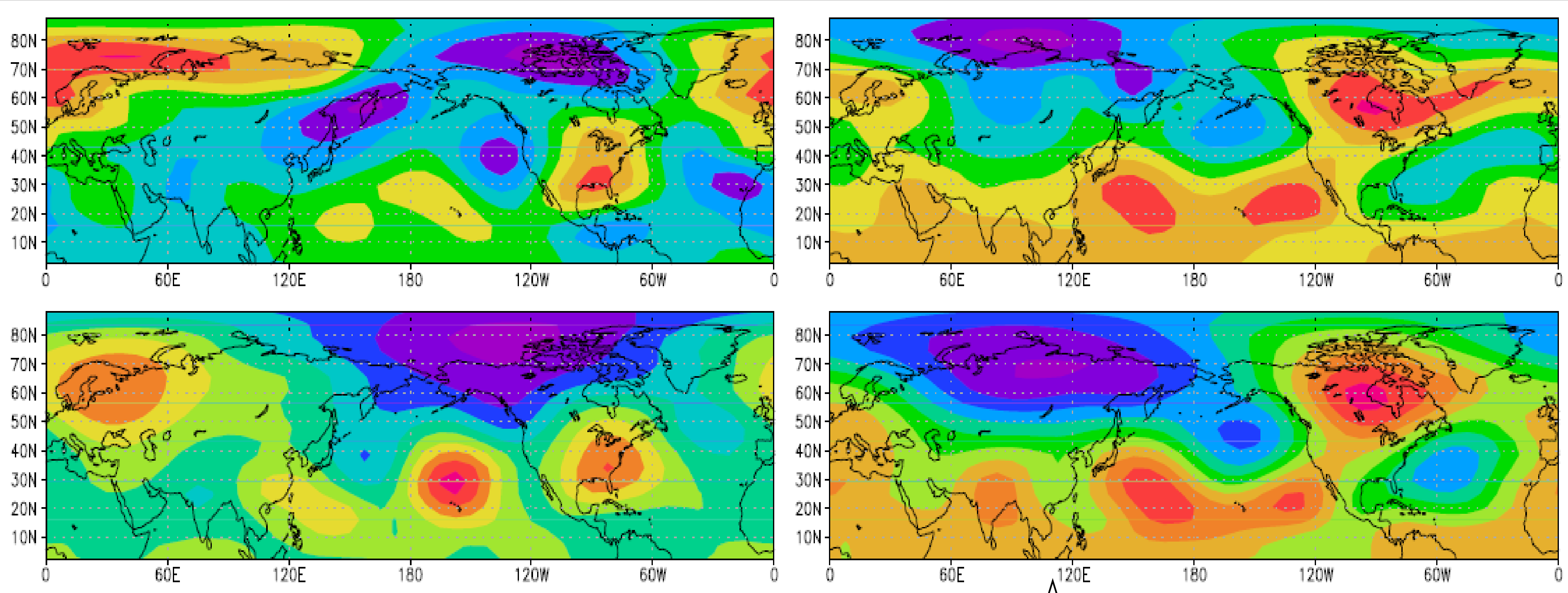
vs

Complex EOF1 (9-16day bandpassed data) (bottom)



Pattern correlation 0.73/0.77

E.V.1 → Complex EOF1 (09-16d bandpassed data) corr: (0.73, 0.77)
E.V.2 → Complex EOF1 (05-09d bandpassed data) corr : (0.69, 0.67)
E.V.3 → Complex EOF1 (02-06d bandpassed data) corr : (0.78, 0.78)
E.V.4 → Complex EOF2 (18-23d bandpassed data) corr : (0.55, 0.44)
E.V.5 → Complex EOF2 (09-16d bandpassed data) corr : (0.39, 0.55)
☆ E.V.6 → Complex EOF1 (18-40d bandpassed data) corr : (0.89, 0.60)



☆ Pattern correlation 0.60/0.89

Normal modes of the system are strongly connected with variability patterns ! Why?

Linear approximation

$$\frac{\partial \Delta \psi}{\partial t} + J(\psi, \Delta \psi + l + H) = -\alpha \Delta \psi + \mu \Delta^2 \psi + f_{ext}.$$

$$\Downarrow$$

$$\frac{\partial h}{\partial t} = -\Delta^{-1} J(h, \Delta \overline{\psi} + l + H) - \Delta^{-1} J(\overline{\psi}, \Delta h) - \alpha h + \mu \Delta h + \Delta^{-1} J(h, \Delta h)$$

$$\frac{\partial h}{\partial t} = Ah + R(h) \qquad R(h) \Rightarrow -\lambda h + \zeta$$

$A - \lambda E$ **- stable!**

ζ **- Gaussian w.noise**

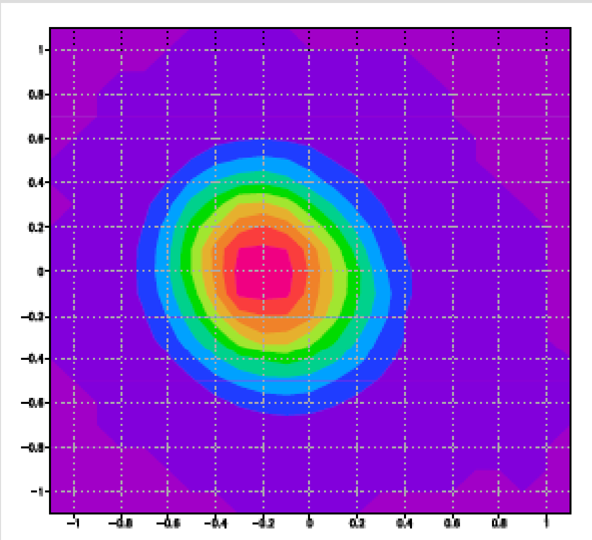
$$\Downarrow$$

$$\frac{\partial h}{\partial t} = (A - \lambda E)h + \zeta$$

$$\frac{\partial h}{\partial t} = Lh + \zeta \qquad L = A - \lambda E$$

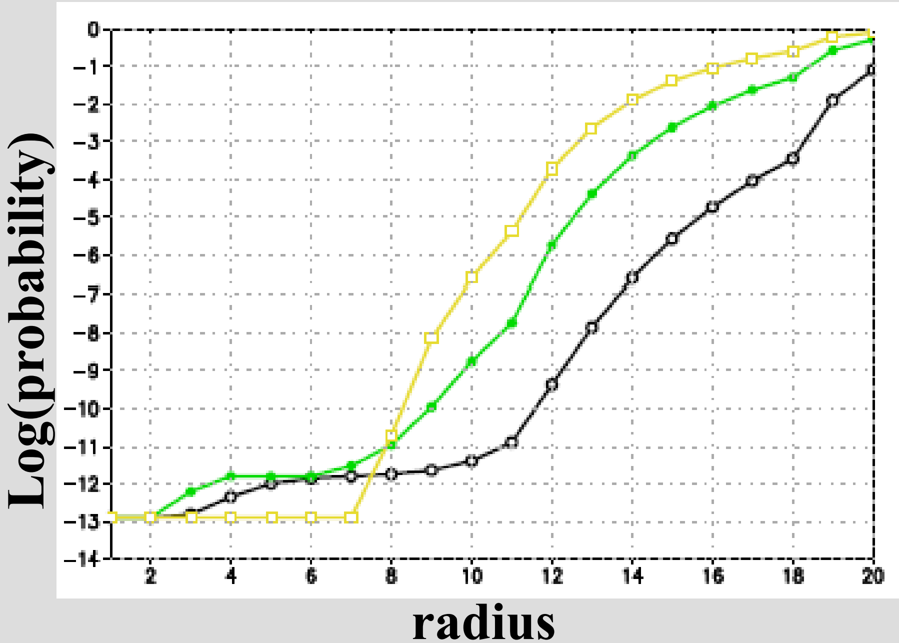
PDF of the linear system is $\rho(\psi) = c_o \exp((-C(0)\psi, \psi))$

1. PDF must be gaussian



PDF of the T21 model projected at CEOF1 plane

2. Maximum density should be at the average state



Probability of the system trajectory to be in a ball around given state: average state (yellow), most visited trajectory point (green), average (black)

$$\frac{\partial h}{\partial t} = Lh + \zeta \quad L = A - \lambda E$$

$$h_\Delta(t) = \int_t^{t+\Delta} h(t')dt' / \Delta$$

Low frequency covariance matrix $C_\Delta = < h_\Delta(t) h_\Delta(t)' >$

Uniform white noise assumption $< \zeta(t)\zeta(t)' >= k\delta(t - t')$

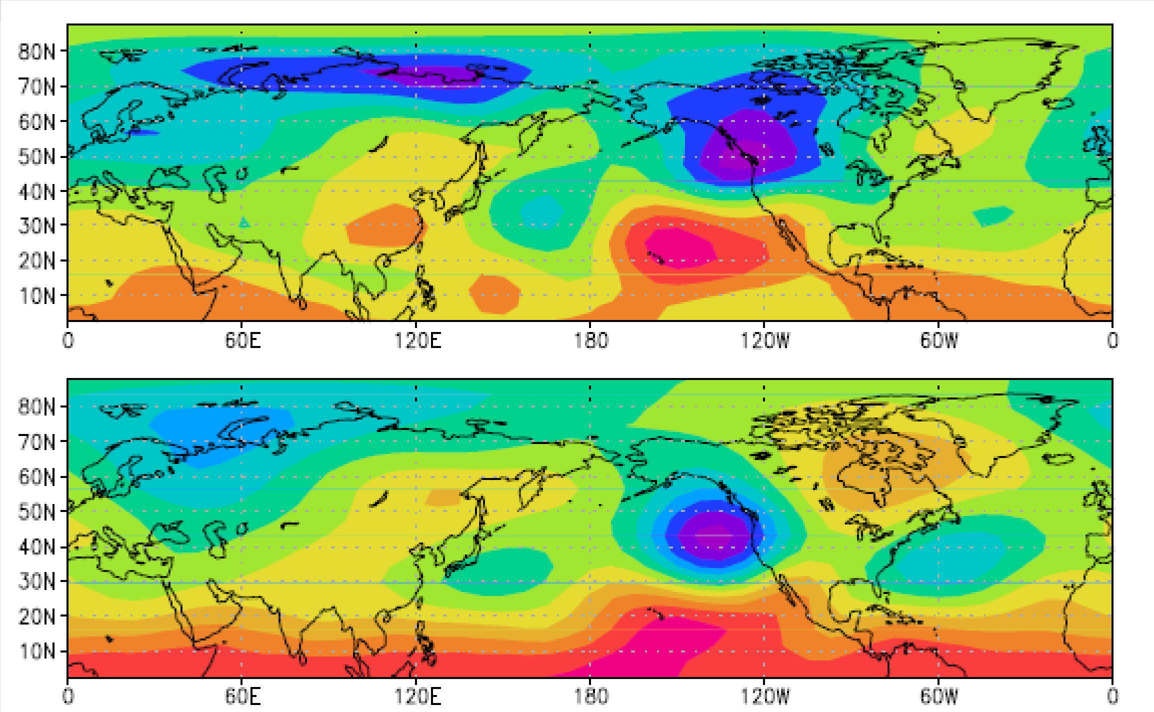
For linear-uniform white noise one can get☆

$$C_\Delta \approx L^{-1} L^{-T} / k \quad \Delta \to \infty$$

Low-frequency EOF are left singular vectors of inversed system linear operator

☆**Dymnikov, Gritsun (Izvestiya, 1999)**

Leading left inv S.V. (top) / Leading low-freq. EOF (50+ d) (bottom)



SVs – Low frequency EOF correlations

	LF1	LF2	LF3	LF4
Sv1	0.84	0.26	0.22	0.28
Sv2	0.48	0.35	0.39	0.52
Sv3	0.11	0.19	0.53	0.38
Sv4	0.09	0.23	0.17	0.42

First low-frequency EOF is reproduced but overall accuracy of "uniform white noise" approximation is not great.

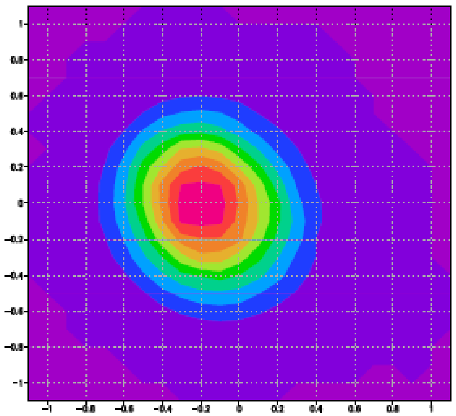
Mean dynamics on leading CEOF1 plane (09-16d bandpassed)

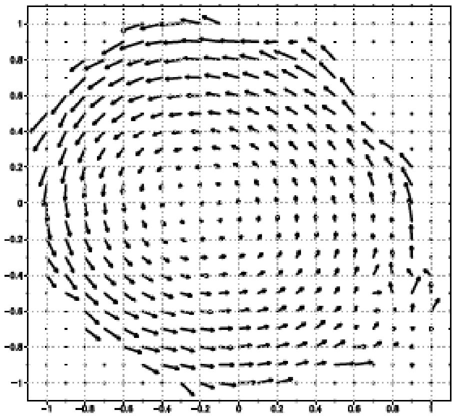
System PDF projected at CEOF1

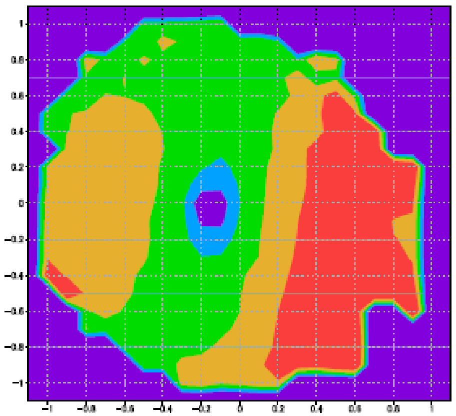
Average tendency projected at CEOF1
$(\Psi(t+\delta)-\Psi(t))/\delta$
0.5d lead time

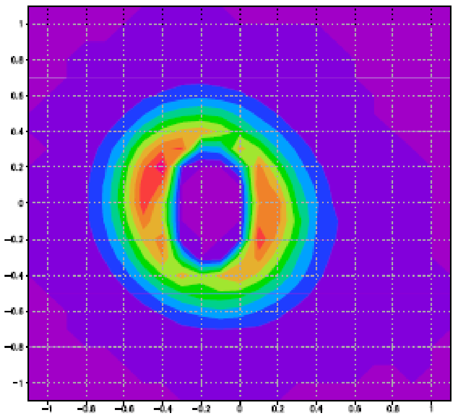
Rotation period (around mean st.)
T_δ
Red > 20days
Orange 16-20 days
Green 12-16 days

PDF of states having at least 30% of tendency along CEOF1







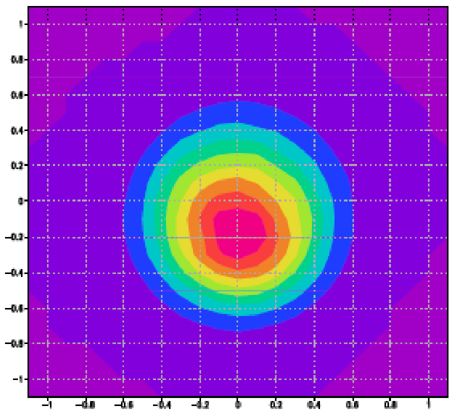


Average tendency suggests rotation around mean state with period 12-20 days, rotation dominates far from the mean state. Nonlinearity (noise?) is not small.

$$T_\delta = 2\pi\delta / a.\sin\{\Psi'_{12}(\delta)\times\Psi'_{12})/|\Psi'_{12}(\delta)\|\Psi'_{12}|\}$$

Mean dynamics on leading CEOF1 plane (05-09d bandpassed)

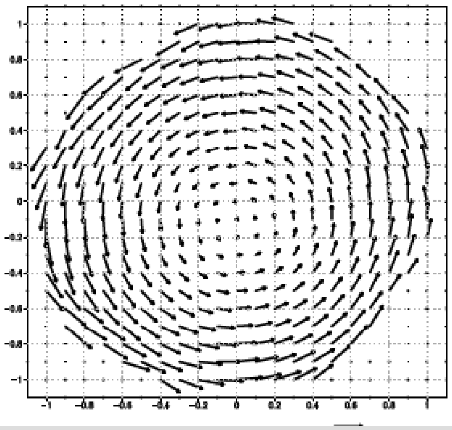
System PDF projected at CEOF1



Average tendency projected at CEOF1

$(\Psi(t+\delta)-\Psi(t))/\delta$

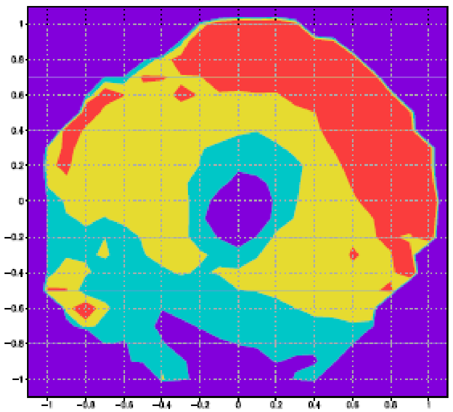
0.5d lead time



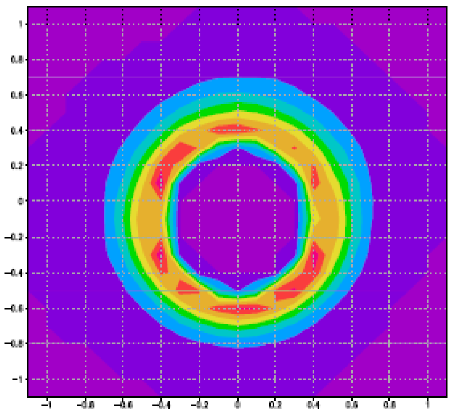
Rotation period (around mean st.)

T_δ

Red > 10days
Yellow 9-10 days
Blue 8-9 days



PDF of states having at least 30% of tendency along CEOF1



Average tendency suggests rotation around mean state with period 9-10 days, rotation dominates far from the mean state. Nonlinearity (noise) is not small.

$$T_\delta = 2\pi\delta / a.\sin\{\Psi'_{12}(\delta)\times\Psi'_{12})/|\Psi'_{12}(\delta)\|\Psi'_{12}|\}$$

Periodic orbits

Periodic orbits are special trajectories of the system returning in the starting position after some T (period)

$$\psi(T) = S(T, \psi(0)) = \psi(0)$$

①Periodic orbits are the part of the system attractor and define recurrent (dynamical) circulation regimes.

②Many chaotic systems have infinite number of periodic orbits.

③ For axiom A and Anosov systems POs are dense on the attractor. Any system characteristic can be approximated by the set of the POs.

④Least unstable orbits are more important!

Methodology for finding orbits

By the definition PO with period T is the system trajectory satisfying

$$\psi(T) = S(T, \psi(0)) = \psi(0)$$

This is the system of nonlinear equations having N+1 unknowns (initial point and period). Phase condition completes the system.

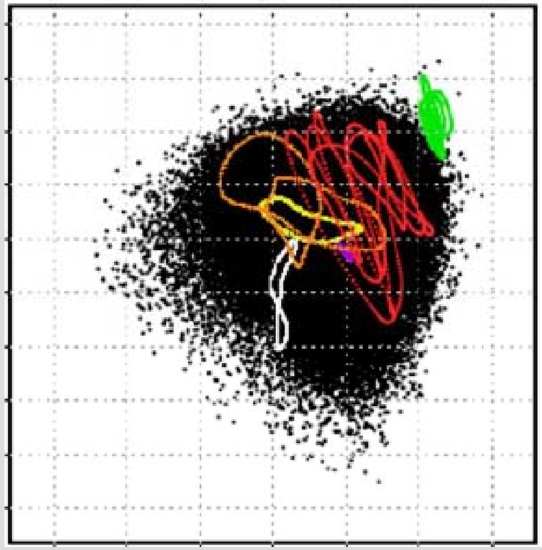
Damped Newton or inexact Quasi-Newton methods could be used to solve the system (+ line search, multi shooting, tenzor correction etc….)

•**Initial guesses for u(0) and *T*** - from the model trajectory {ψ(k), k=1..K} as local minimizers for $|\psi(k+T)-\psi(k)|^2$

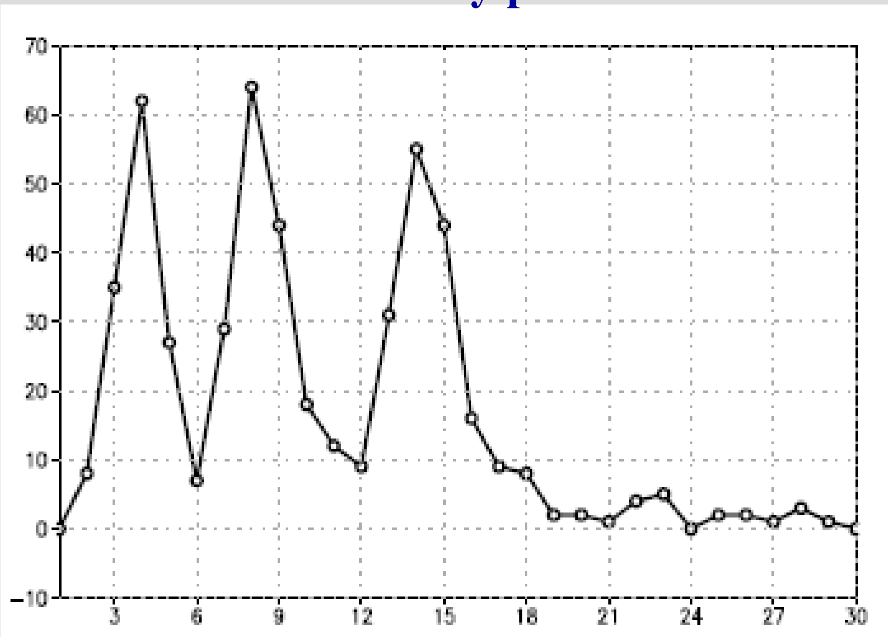
•**What is required?** Tangent model in full space (Newton) Or Tangent and adjoint tangent models in Krylov space (inexact methods). One iteration for finding UPO with period T means one run of tangent (adjoint tangent model) in full (Krylov) space for time T.

UPOs of the T21 system

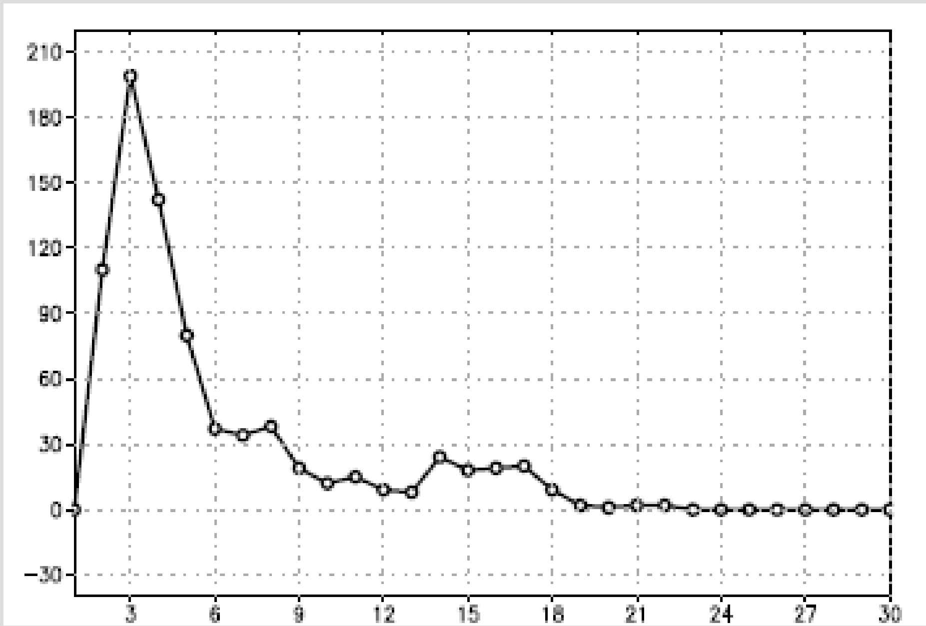
- ☆ 1500+ periodic orbits for T21 system were found
- ☆ All are unstable (25-60+ unstable directions), periods are 2-30+ days
- ☆ UPOs approximates system statistics as predicted!



Distribution of 500 least unstable UPOs by period



Distribution of 800 most unstable UPOs by period



☆ Gritsun (Phil. Trans. R. Soc. A , 2013)

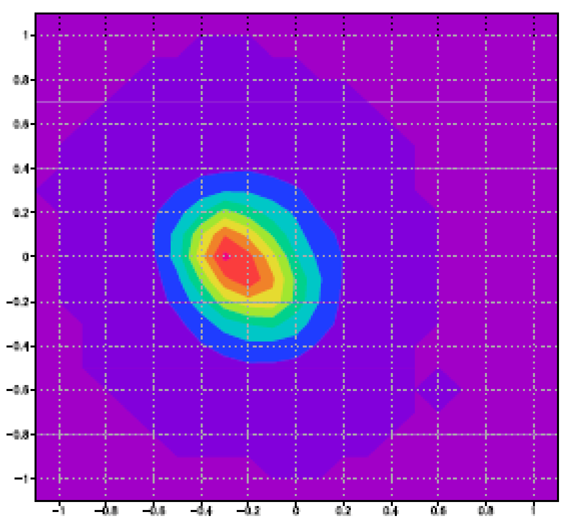
UPOs on leading CEOF1 plane (09-16d bandpassed)

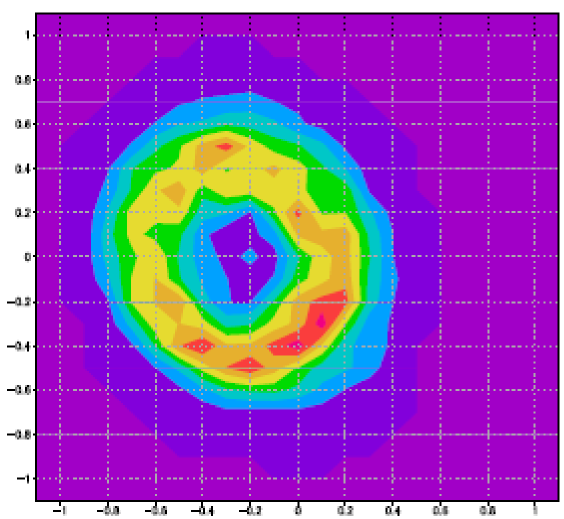
ALL UPOs
PDF at
CEOF1

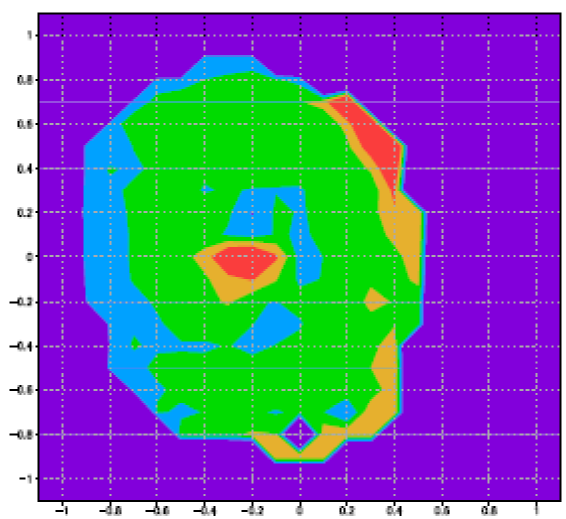
UPOs having at
least 60% of the
tendency
along CEOF1

Rotation period
(around mean st.)

Orange > 16days
Green 14-16 days
Blue 12-14 days







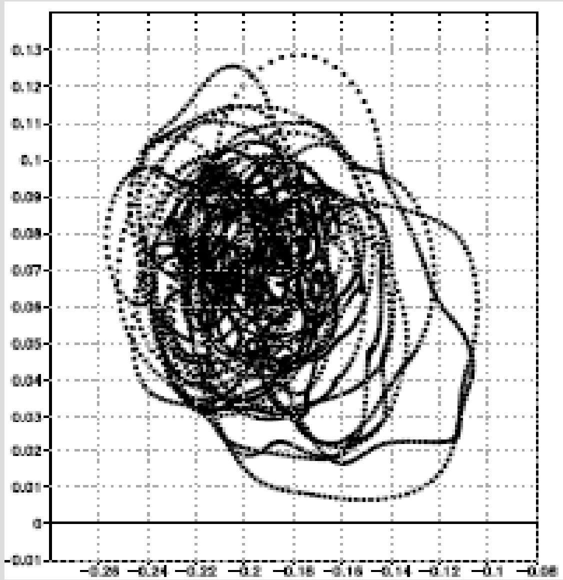
Spatial structure of leading UPOs matches structure characteristic time scale of CEOF1 and leading pair of normal modes.

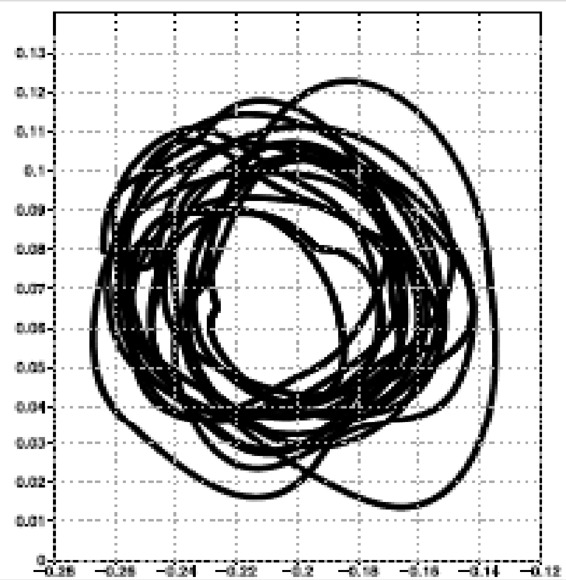
UPOs on leading CEOF1 plane (09-16d bandpassed)

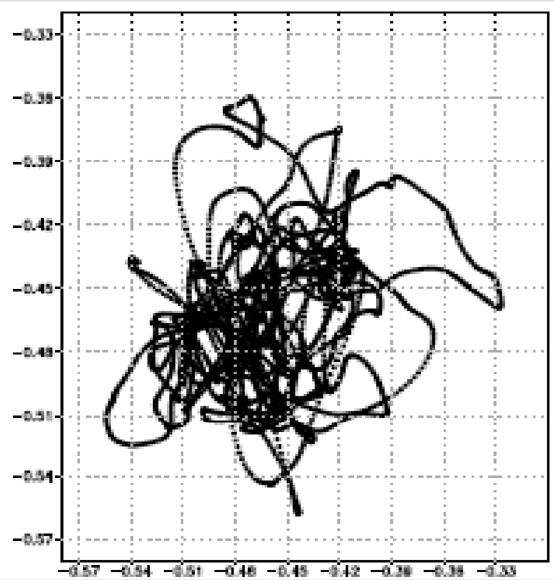
System trajectory (5000d) at CEOF1 (when 60% of the tendency is at CEOF1 plane)

UPOs having at least 60% of the tendency along CEOF1

System trajectory at EOF1-EOF2 (when 60% of the tendency is at EOF1-EOF2 plane)







System shows regular rotational behavior when moving along CEOF1 plane. This is not always the case.

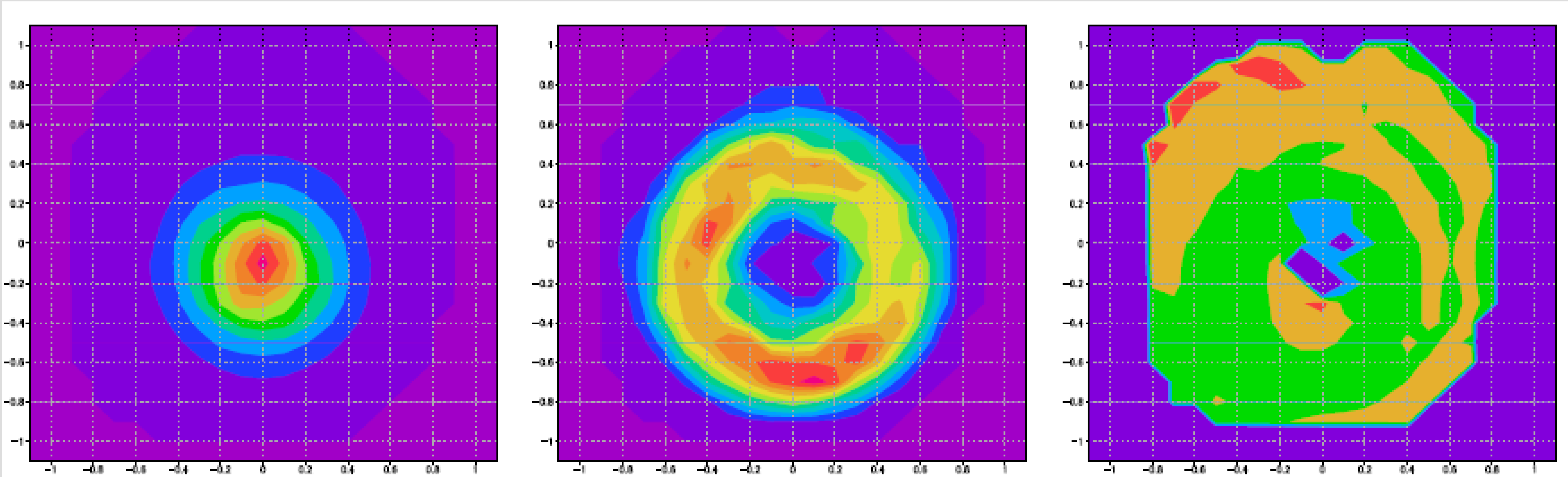
UPOs on leading CEOF1 plane (05-9d bandpassed)

ALL UPOs
PDF at
CEOF1

UPOs having at
least 60% of the
tendency
along CEOF1

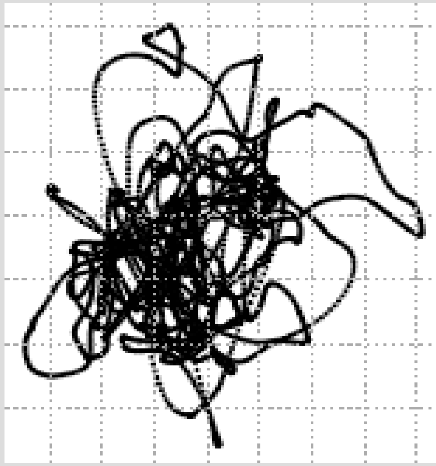
Rotation period
(around mean st.)

Orange 8- 9days
Green 7-8 days
Blue <7 days



Spatial structure of leading UPOs matches structure characteristic time scale of CEOF1 and second leading pair of normal modes.

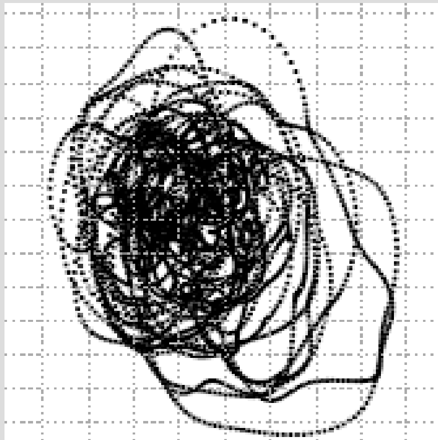
Summary



1. Normal modes explain some patterns (leading CEOFs) of the system variability and corresponding timescales.

2. It seems that linear approximation may not explain why. “Linear” behavior of the system occurs far from the mean state.

3. Least unstable POs of the system are important. Groups of UPOs are positioned along corresponding leading CEOFs (and normal modes). Periods of UPOs match normal mode frequencies. System moves along UPOs showing regular behavior.



4. Phase portrait of the system remembers its stable state analog?

$$h_\Delta(t) = \int_t^{t+\Delta} h(t')dt'/\Delta \quad C_\Delta = < h_\Delta(t) h_\Delta(t)' >$$

$$C(\Delta) = < h(t+\Delta) h(t)' > \quad \overline{C} = \int_0^{+\infty} C(\Delta) d\Delta$$

$$C_\Delta \approx (\overline{C} + \overline{C}^T)/\Delta \quad \Delta \to \infty$$

Lyapunov equation

$$LC(0) + C(0)L^T = kE \qquad C(\Delta) = \exp(L\Delta)C(0)$$

$$C(0)L^{-T} + L^{-1}C(0) = kL^{-1}L^{-T}$$

$$\overline{C} = L^{-1}C(0)$$

$$\overline{C} + \overline{C} = kL^{-1}L^{-T}$$