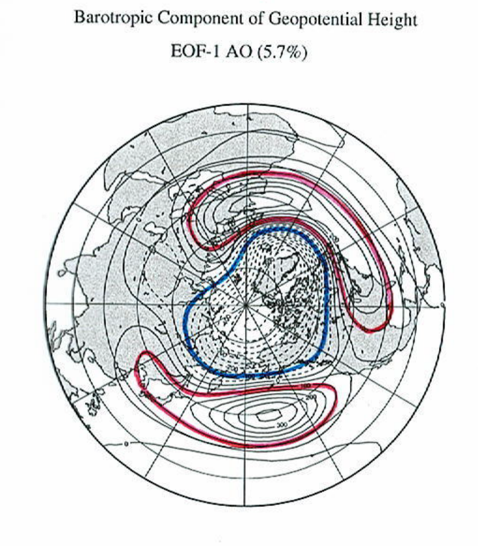
3D Spectral Energetics Analysis and Rossby Wave Saturation Theory



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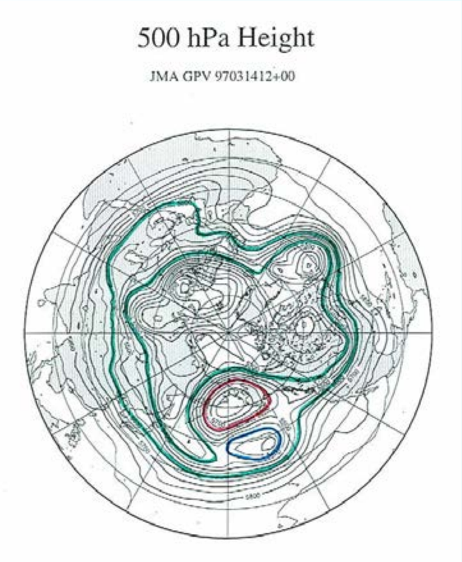
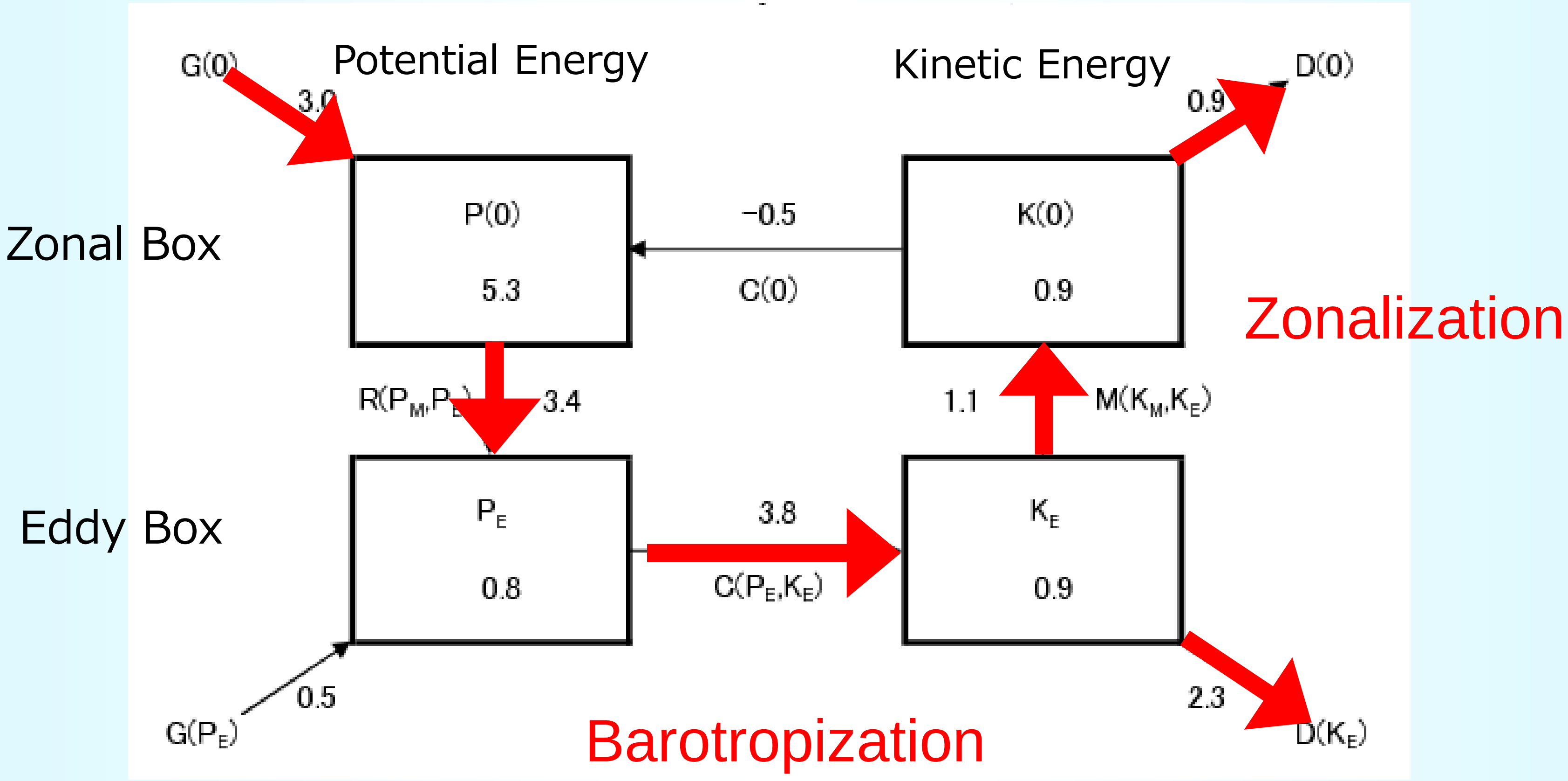


(Presentation at MODES 2015)

Lorenz Energy Box Diagram

Energy accumulates at zonal wave 0



Energy accumulates at vertical wave 0

(Kung and Tanaka 1983, JAS)

Lorenz cycle, Saltzman cycle

(Saltzman 1957; 1970)

$$p = \sum_{n=-\infty}^{\infty} p_n \exp(inx)$$

K(0)

$$\frac{\partial Kz}{\partial t} = \sum_{n=1}^{N} M(n) + C(0) - D(0),$$

K(n)

$$\frac{\partial K(n)}{\partial t} = -M(n) + L(n) + C(n) - D(n), \quad n = 1, 2, 3, \cdots$$

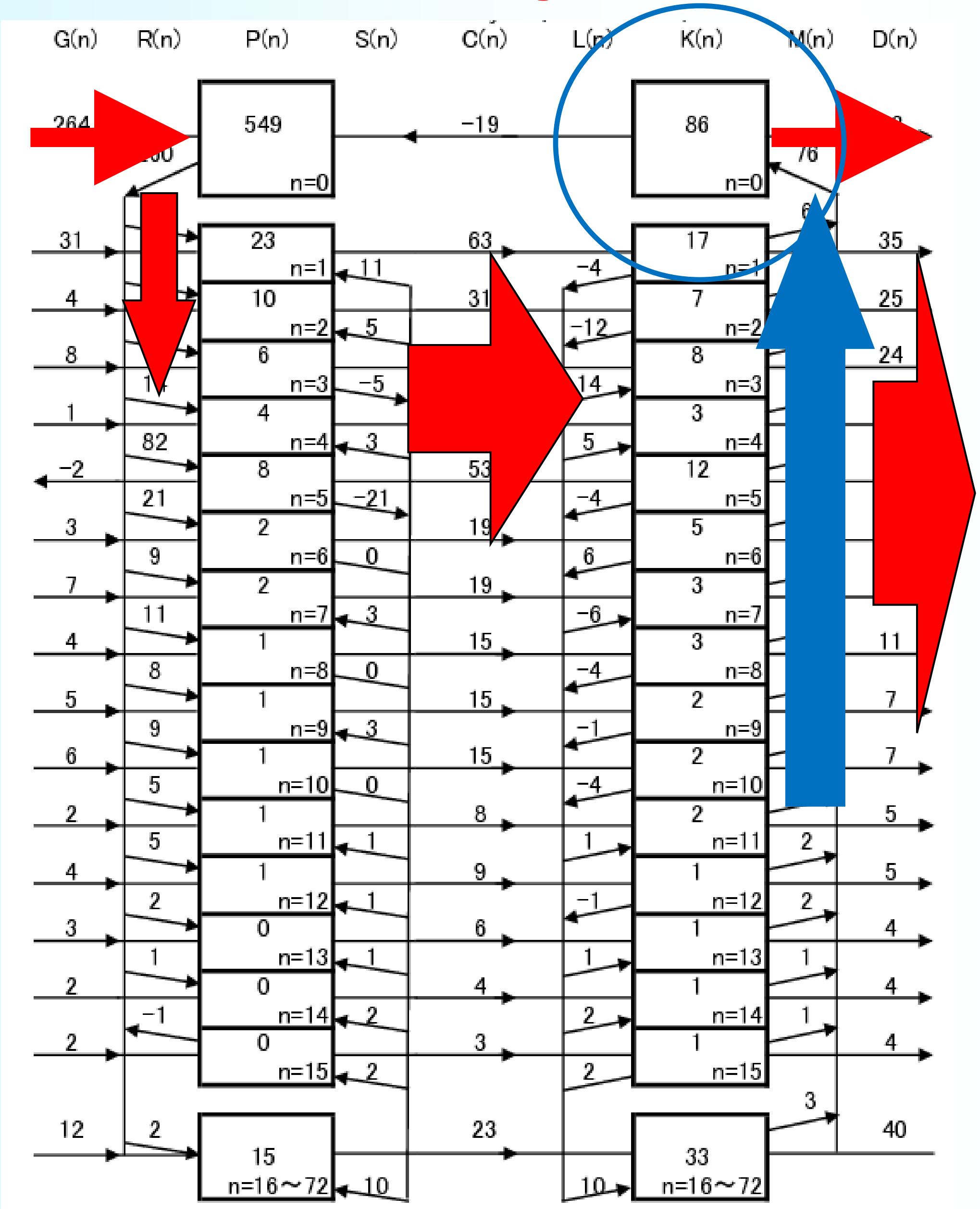
P(0)

$$\frac{\partial Pz}{\partial t} = -\sum_{n=1}^{N} R(n) - C(0) + G(0),$$

P(n)

$$\frac{\partial P(n)}{\partial t} = R(n) + S(n) - C(n) + G(n), \quad n = 1, 2, 3, \cdots$$

Saltzman cycle



G: Generation of P(n)

P: Available potential energy

R: zonal-wave interaction of P(n)

S: wave-wave interaction of P(n)

C: Baroclinic conversion
from P(n) to K(n)

K: Kinetic energy

M: zonal-wave interaction of K(n)

L: wave-wave interaction of K(n)

D: Dissipation of K(n)

(Saltzman 1957 & 1970)

(Kung and Tanaka 1983 &1984)

Data

- JRA-25

JRA-25 Reanalysis by JMA

  - $2.5^\circ \times 2.5^\circ$, 23 levels (1000 - 0.4 hPa)
- NCEP/NCAR reanalysis

NOAA CDC

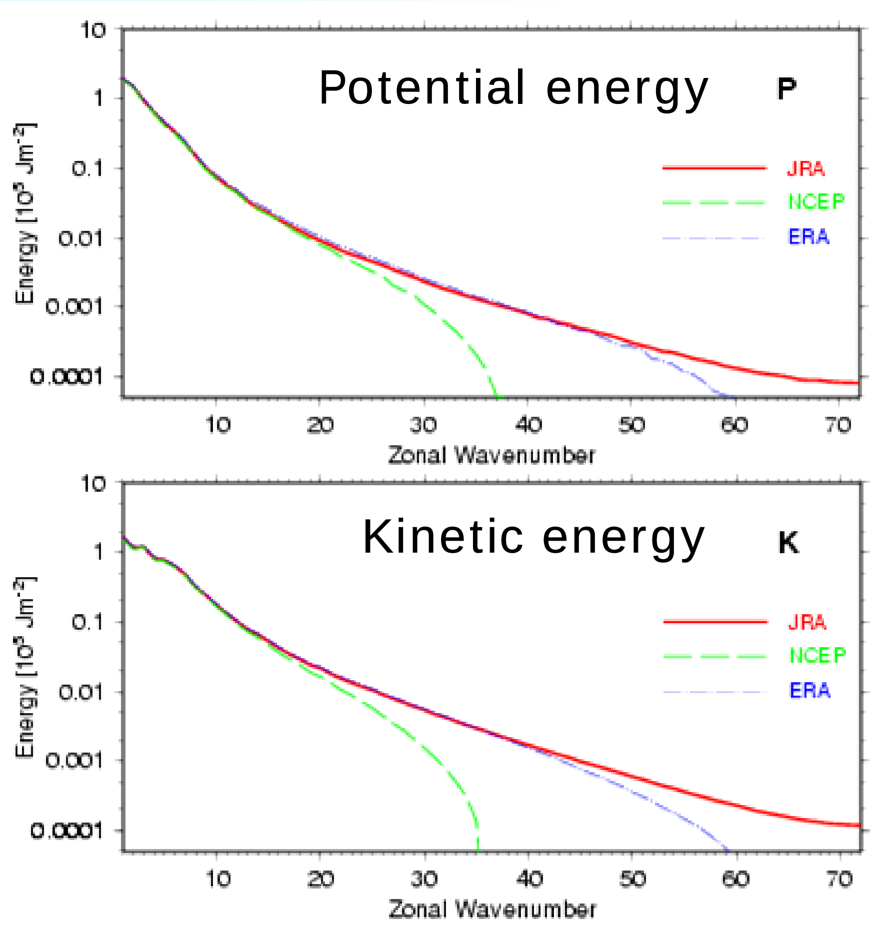
  - $2.5^\circ \times 2.5^\circ$, 17 levels (1000 - 10 hPa)
- ERA-40

ECMWF

  - $2.5^\circ \times 2.5^\circ$, 23 levels (1000 - 1 hPa)

- 1990/91 DJF (3 Month)
- u, v, T, q

Available Potential Energy P(n)
Kinetic energy K(n)



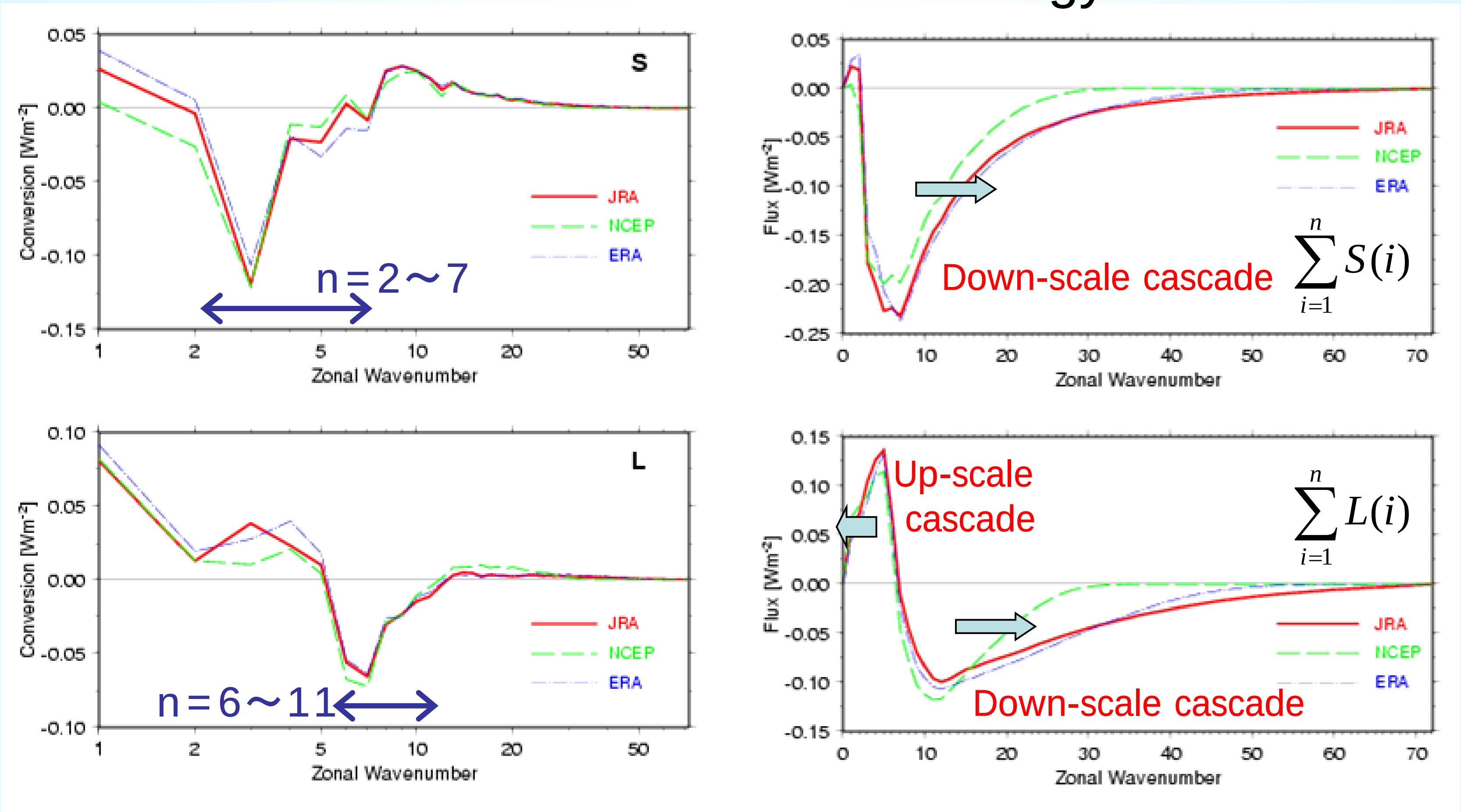
- NCEP up to n=35
- ERA up to n=60

total energy [10^4 Jm^{-2}]

	NCEP	JRA	ERA
P_E	64	67	70
K_E	77	82	84

Wave-wave interactions (S, L)

Energy Flux



2D Spectral model

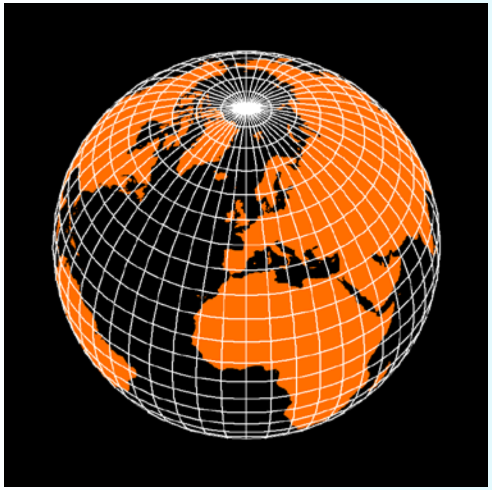
- 1D: Expansion in Fourier harmonics
- 2D: Expansion in spherical harmonics

$$p = \sum_{n=-\infty}^{\infty} p_n \exp(inx)$$

$$\frac{\partial p}{\partial x} = \sum_{n=-\infty}^{\infty} in\, p_n \exp(inx)$$

$$Y_l^n(\lambda, \theta) = P_l^n(\theta) \exp(in\lambda)$$

$$p(\lambda, \theta) = \sum_{n=-N}^{N} \sum_{l=|n|}^{L} p_{nl} Y_l^n(\lambda, \theta)$$



3D Spectral model

- Vertical normal mode
- Horizontal normal mode: Hough harmonics

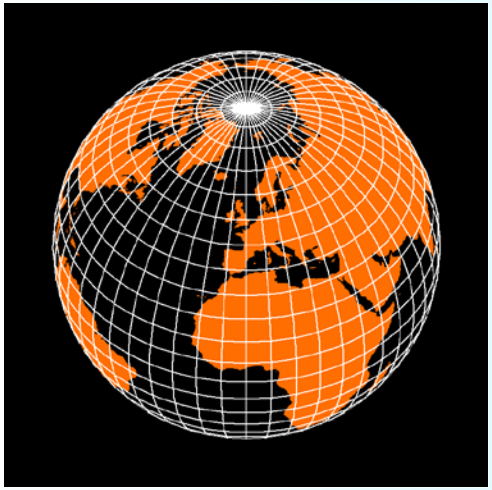
Expansion in 3D Normal Mode Functions

$$\Pi_{nlm}(\lambda,\theta,\sigma)=\Theta_{nlm}(\theta)\,\mathrm{G}_{\mathrm{m}}(\sigma)\exp(in\lambda)$$

$$U(\lambda,\theta,\sigma)=\sum_{n=-N}^{N}\sum_{l=0}^{L}\sum_{m=0}^{M} w_{nlm}X_m\Pi_{nlm}(\lambda,\theta,\sigma)$$

$$\frac{dw_i}{d\tau}=-i\sigma_i w_i-i\sum_{jk} r_{ijk}w_j w_k+f_i,\quad i=1,2,3,\ldots$$

$$E_i=\frac{1}{2}p_s h_m|w_i|^2,\quad w_{nlm}\to w_i$$



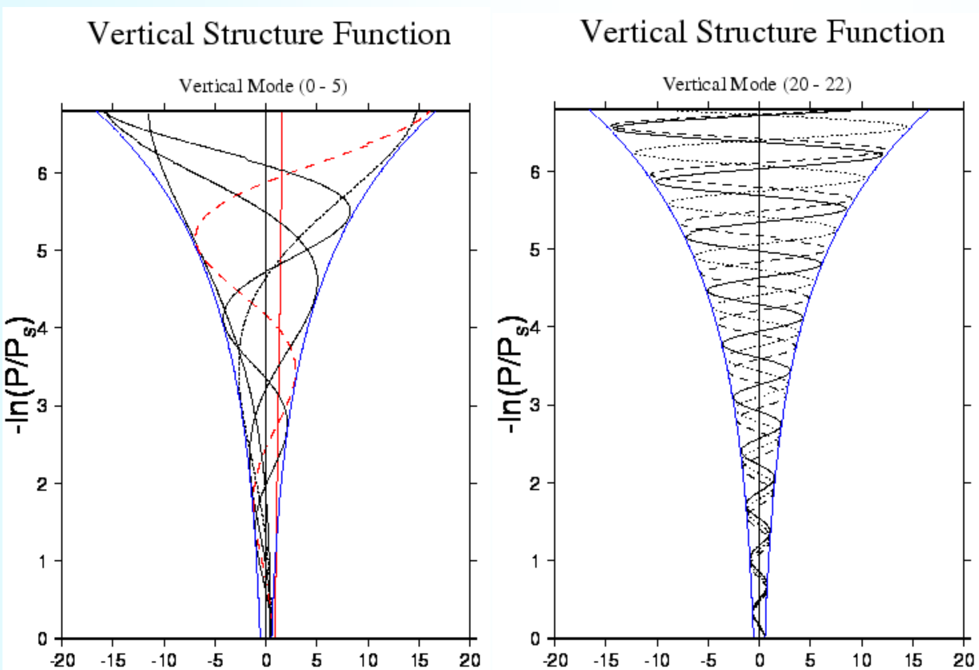
Vertical energy spectrum

Vertical modes
$$\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial G_m}{\partial\sigma}\right)+\lambda_m G_m=0,\qquad \lambda_{\mathrm{m}}=\frac{R\gamma}{gh_m}$$

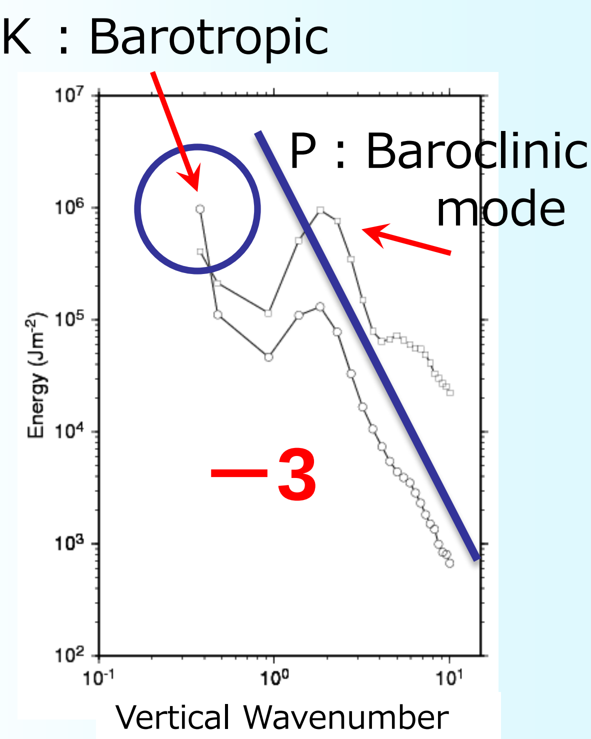
$$G_0(\sigma)=C_1\sigma^{r_1}+C_2\sigma^{r_2}$$

Barotropic and baroclinic modes

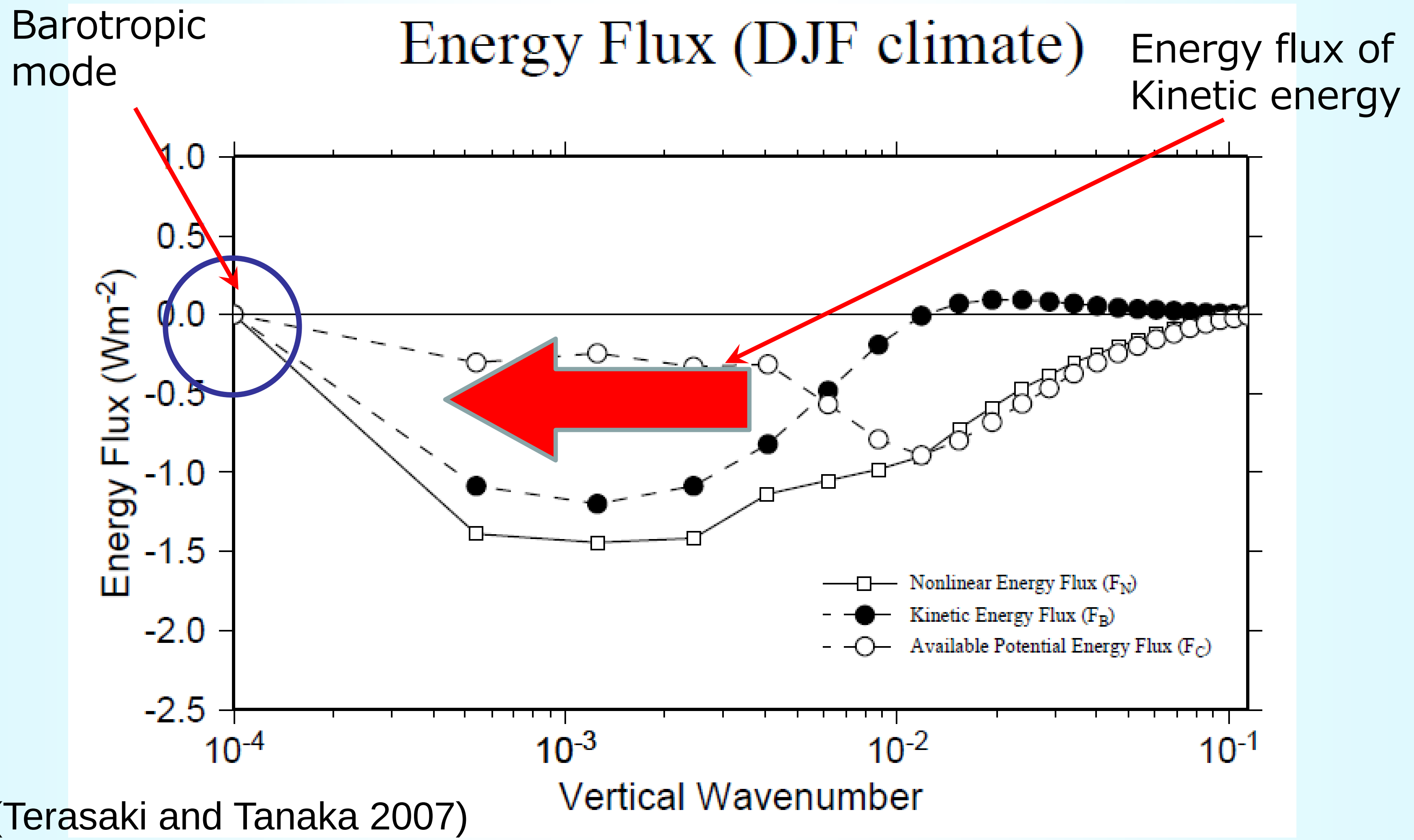
$$G_m(\sigma)=\sigma^{-\frac{1}{2}}\left(C_1\sin(\mu\ln\sigma)+C_2\cos(\mu\ln\sigma)\right)\qquad \mu=\sqrt{\lambda_m-\frac{1}{4}}$$



Terasaki and Tanaka (2007)



Barotropization
by baroclinic instability



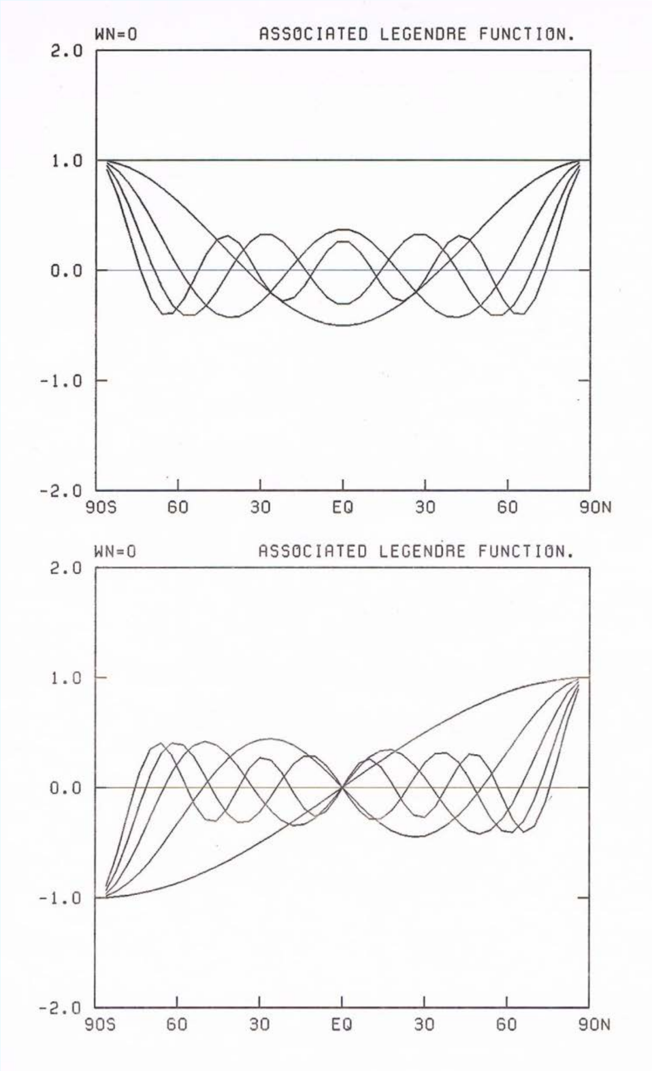
(Terasaki and Tanaka 2007)

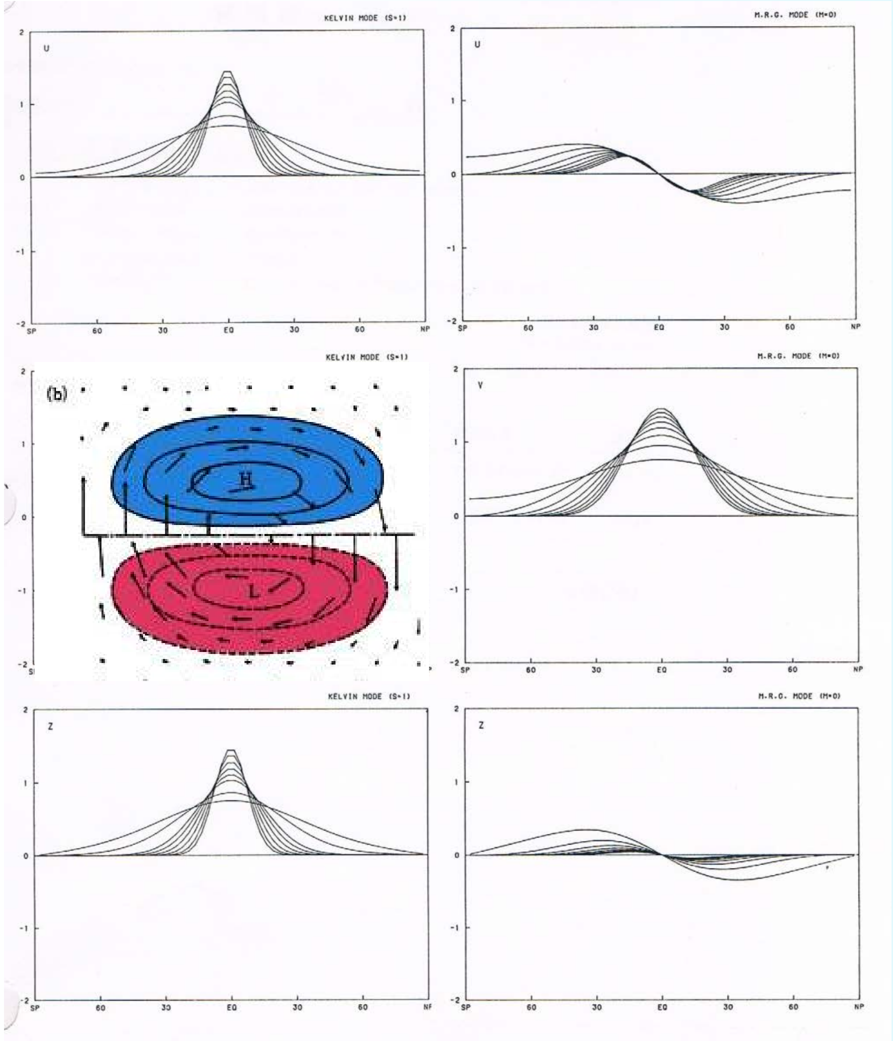
Spherical harmonics (n=0)

Hough harmonics

Kelvin mode

Mixed Rossby-gravity mode





Zonal Energy Spectrum n

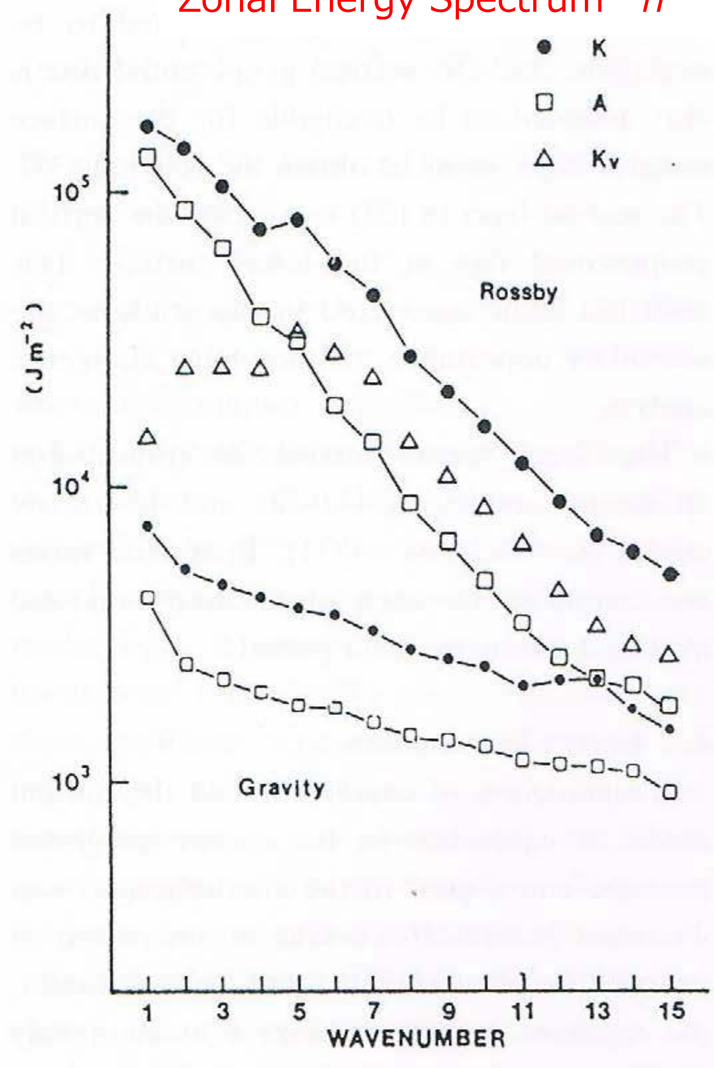


Fig. 2. Energy distributions in the wavenumber domain. K : kinetic energy, A : available potential energy, K_v : v-component of K

Vertical Energy Spectrum m

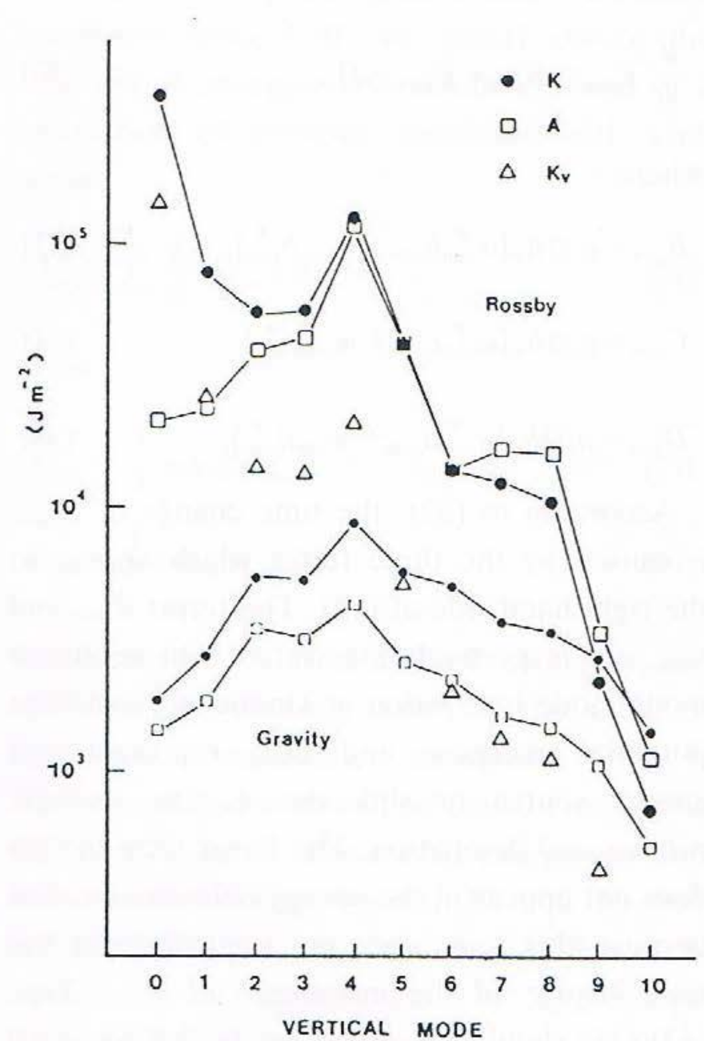
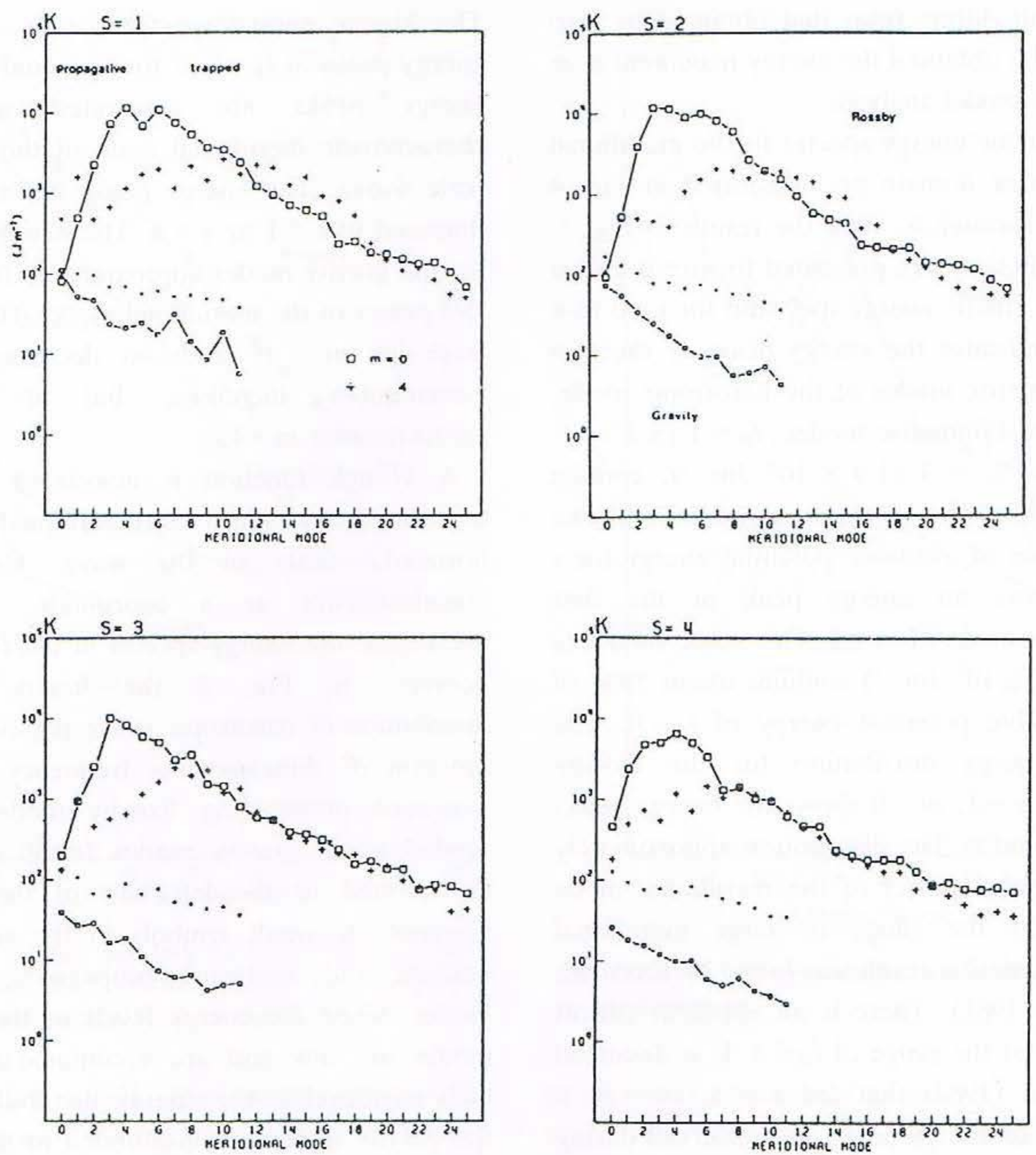


Fig. 3. Eddy energy distributions in the vertical mode domain.

Tanaka (1985)

Meridional Energy Spectrum Ⅰ



Tanaka (1985)

Energy spectrum in the 3D wavenumber space

$$c = -\frac{\beta}{n^2 + l^2 + m^2} = -\frac{\beta}{k^2}$$

n, l, m : zonal, meridional and vertical waves

k : total wave

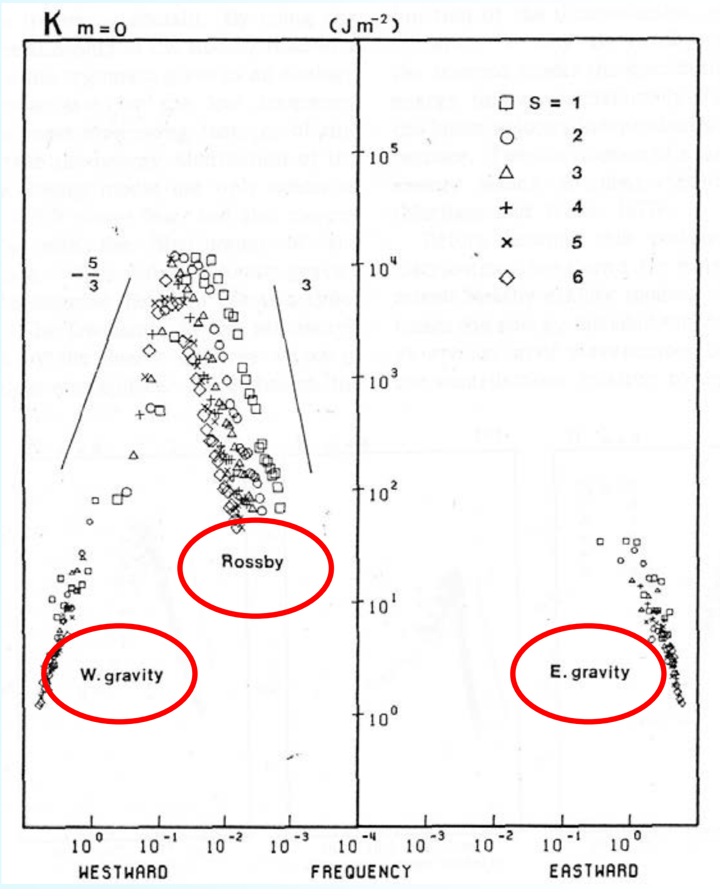
$$c = \sigma / n$$

$$\frac{dw_i}{d\tau} = -i\sigma_i w_i - i\sum_{jk} r_{ijk} w_j w_k + f_i$$

Use c for the scale in place of 3D wavenumber

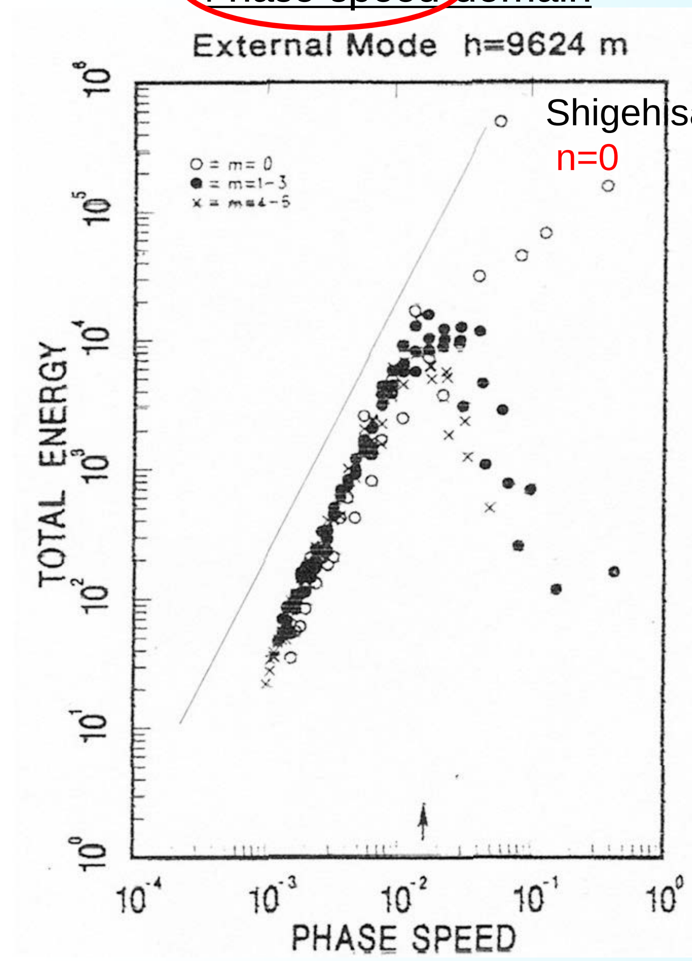
Observed energy spectrum in *c*-domain

Frequency domain



Tanaka (1985)

Phase speed domain



Shigehisa mode

n=0

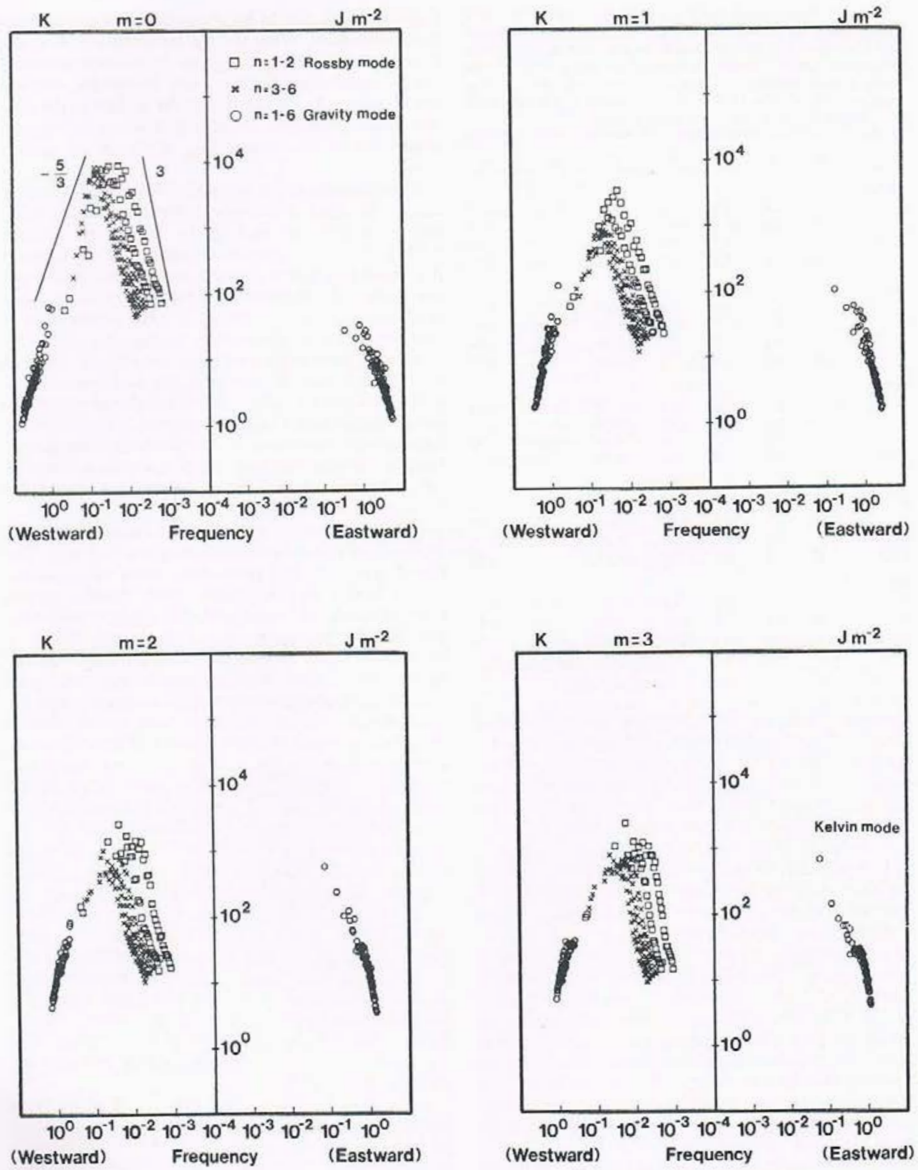
Tanaka and Kasahara (1992)

Frequency domain

Tanaka and Kimura (1996)

Vertical mode
m=0

Vertical mode
m=1

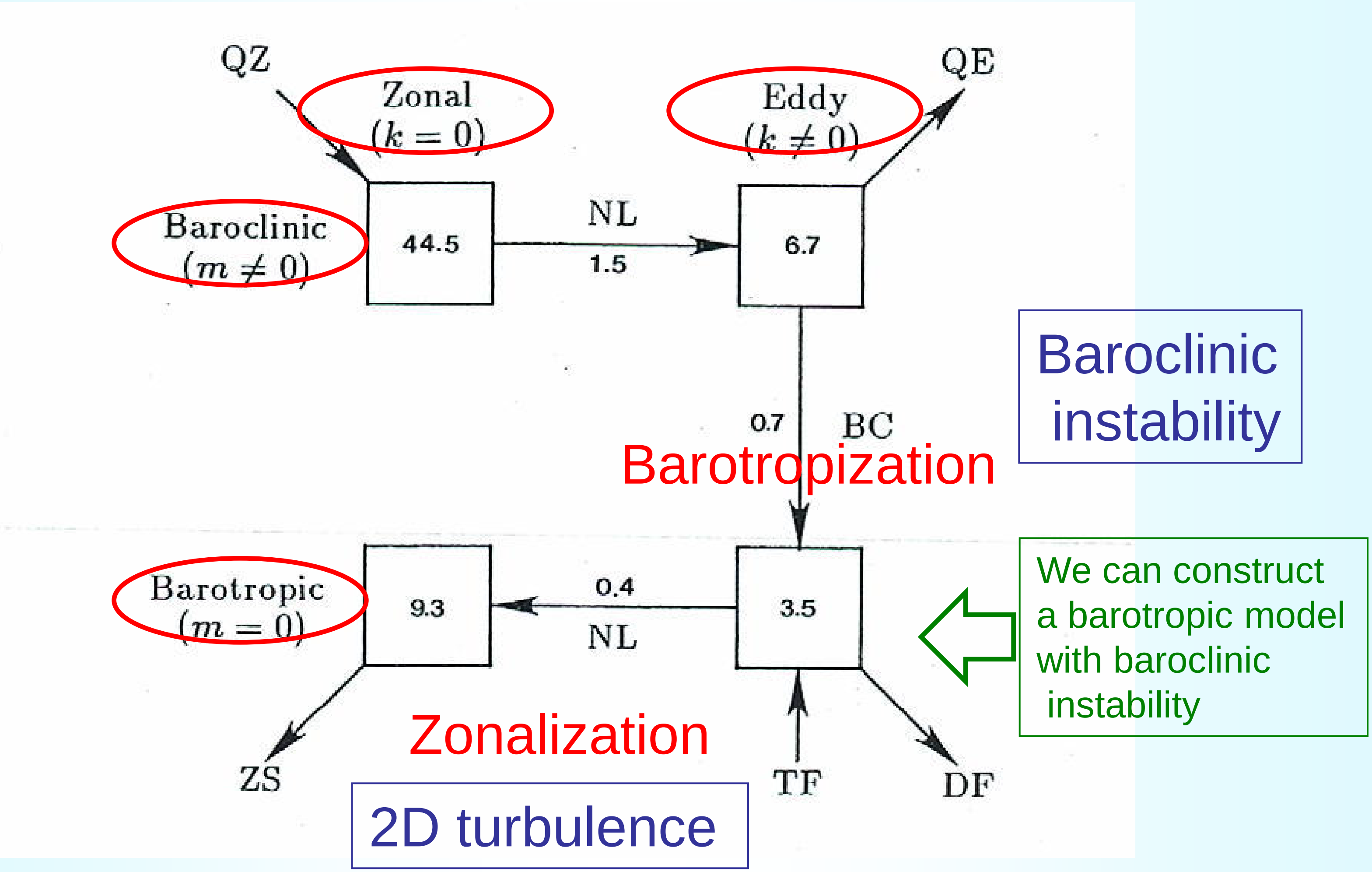


Vertical mode
m=2

Vertical mode
m=3

FIG. 3. Kinetic energy spectra in the dimensionless frequency domain (normalized by 2Ω) for vertical indices m = 0–3. The energy of the Rossby modes and gravity modes are plotted for wavenumber n = 1–6. Note that the frequency on the abscissa is the eigenfrequency of Laplace's tidal equation rather than the analyzed wave frequency in the space–time spectra.

Energy flow box diagram
in barotropic-baroclinic decomposition



Numerical simulations of blocking and AO

Primitive Equation Model

Primitive Equation Model

$$\mathbf{M}\frac{\partial U}{\partial t} + \mathbf{L}\,U = N + F, \qquad (1)$$

where

$$U = (u,\, v,\, \phi')^T, \qquad (2)$$

$$\mathbf{M} = diag(1,\, 1,\, -\frac{\partial}{\partial p}\frac{p^2}{R\gamma}\frac{\partial}{\partial p}), \qquad (3)$$

$$\mathbf{L} = \begin{pmatrix} 0 & -2\Omega\, sin\theta & \frac{1}{a\, cos\theta}\frac{\partial}{\partial\lambda} \\ 2\Omega\, sin\theta & 0 & \frac{1}{a}\frac{\partial}{\partial\theta} \\ \frac{1}{a\, cos\theta}\frac{\partial}{\partial\lambda} & \frac{1}{a\, cos\theta}\frac{\partial()cos\,\theta}{\partial\theta} & 0 \end{pmatrix}, \qquad (4)$$

$$N = \begin{pmatrix} -V\cdot\nabla u - \omega\frac{\partial u}{\partial p} + \frac{tan\theta}{a}uv \\ -V\cdot\nabla v - \omega\frac{\partial v}{\partial p} - \frac{tan\theta}{a}uu \\ \frac{\partial}{\partial p}(\frac{p^2}{R\gamma}V\cdot\nabla\frac{\partial\phi}{\partial p} + \omega p(\frac{p}{R\gamma}\frac{\partial\phi}{\partial p})) \end{pmatrix}, \qquad (5)$$

$$F = (F_u,\, F_v,\, \frac{\partial}{\partial p}(\frac{pQ}{C_p\gamma}))^T. \qquad (6)$$

Barotropic Component of the Atmosphere

○ **Vertical Transform**

$$(u, v, \phi')_0^T = \frac{1}{p_s}\int_0^{p_s}(u, v, \phi')^T G_0 dp \qquad (1)$$

○ **Barotropic Model**

$$\frac{\partial u}{\partial t} = -\vec{v}\cdot\nabla u + fv - \frac{\partial \phi}{\partial x} + F_x \qquad (2)$$

$$\frac{\partial v}{\partial t} = -\vec{v}\cdot\nabla v - fu - \frac{\partial \phi}{\partial y} + F_y \qquad (3)$$

$$\frac{\partial \phi}{\partial t} = -\vec{v}\cdot\nabla \phi - \bar{\phi}\nabla\cdot\vec{v} + F_z \qquad (4)$$

○ **3-D Spectral Transform**

$$U(\lambda, \theta, p, t) = \sum_{nlm} w_{nlm}(t) X_m \Pi_{nlm}(\lambda, \theta, p), \qquad (5)$$

$$w_{nlm}(t) = < U(\lambda, \theta, p, t), X_m^{-1}\Pi_{nlm} > \qquad (6)$$

where $U(\lambda, \theta, p, t) = (u, v, \phi')^T$, $w_{nlm}(t)$ is the spectral expansion coefficient, $X_m = diag(c_m, c_m, c_m^2)$, and Π_{nlm} is the 3-D NMF.

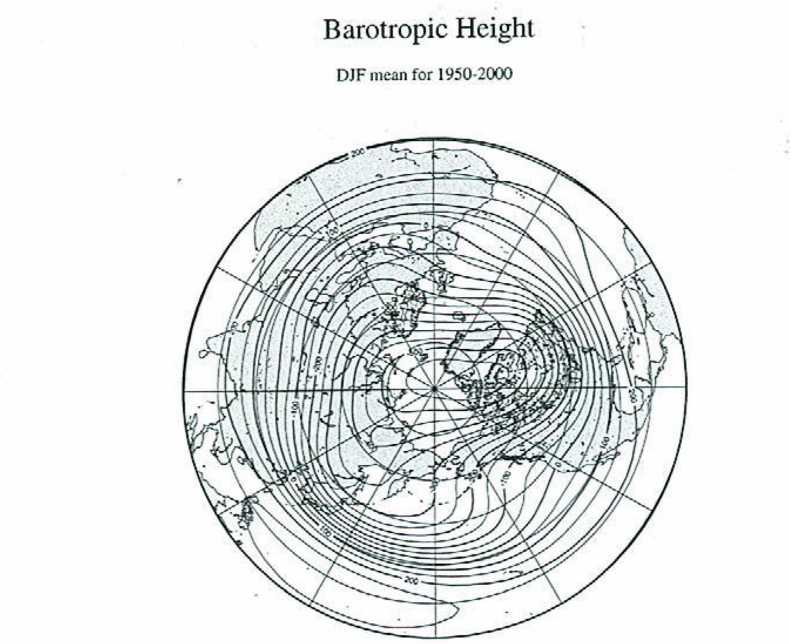
Numerical simulations of blocking and AO

$$\frac{dw_i}{d\tau} = -i\sigma_i w_i - i\sum_{jk} r_{ijk} w_j w_k + f_i$$

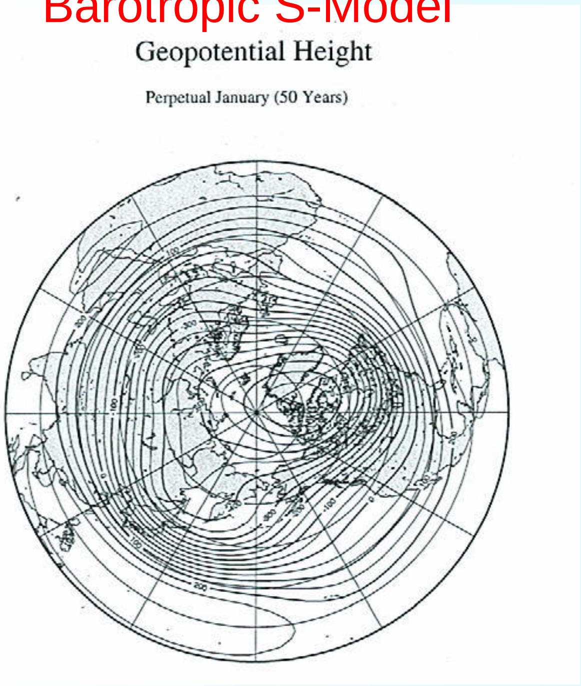
$$E_i = \frac{1}{2} p_s h_m |w_i|^2$$

Barotropic S-Model
(Tanaka 2003, JAS)

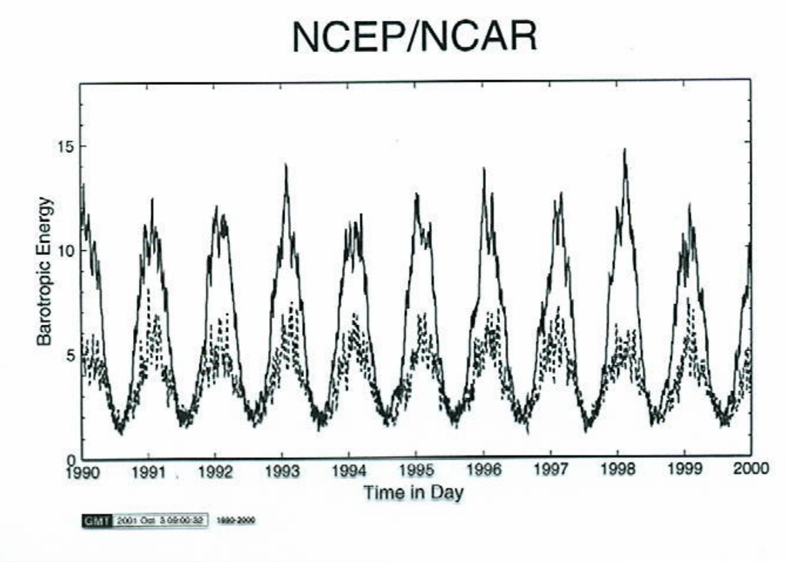
NCEP/NCAR



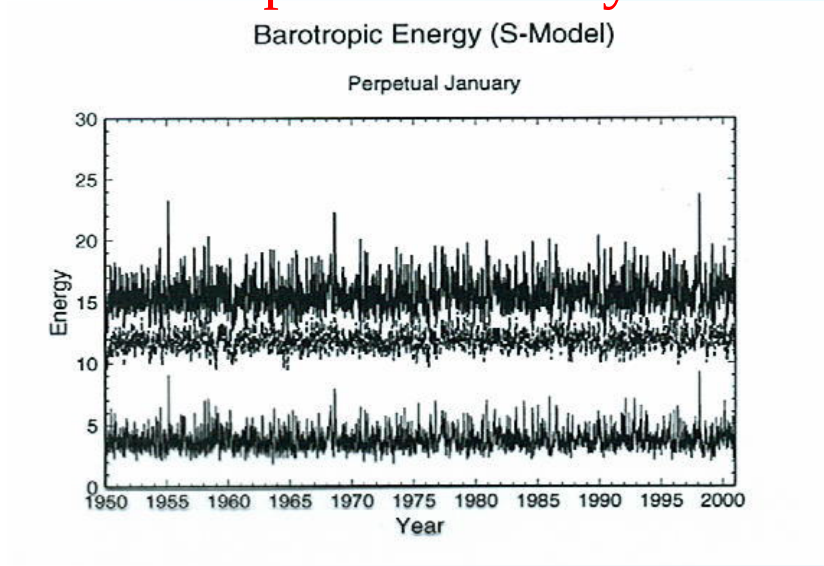
Barotropic S-Model



Barotropic energy



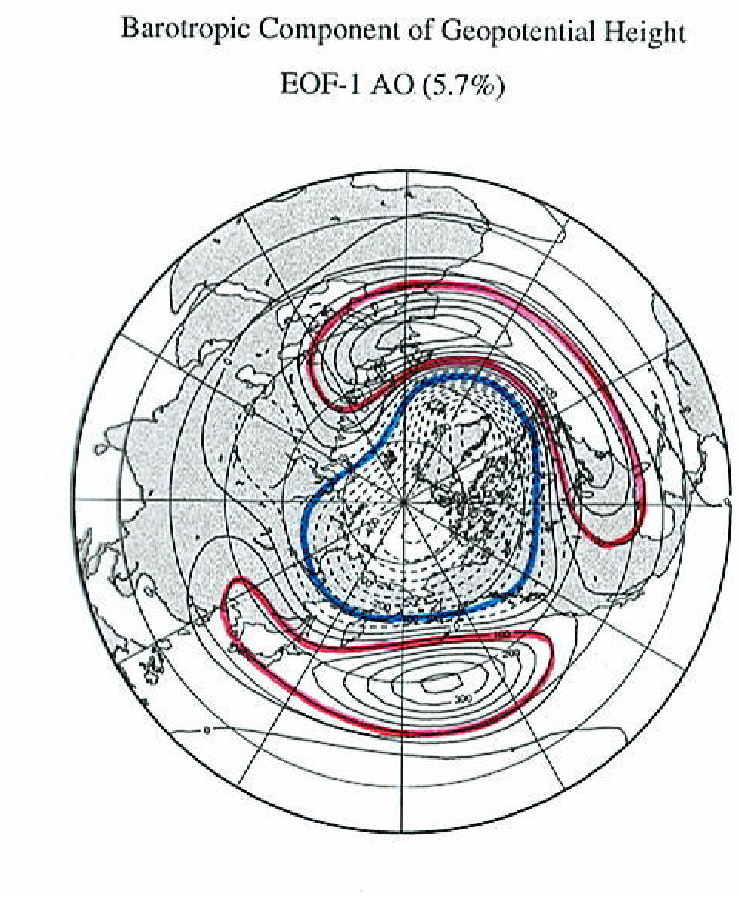
Perpetual January



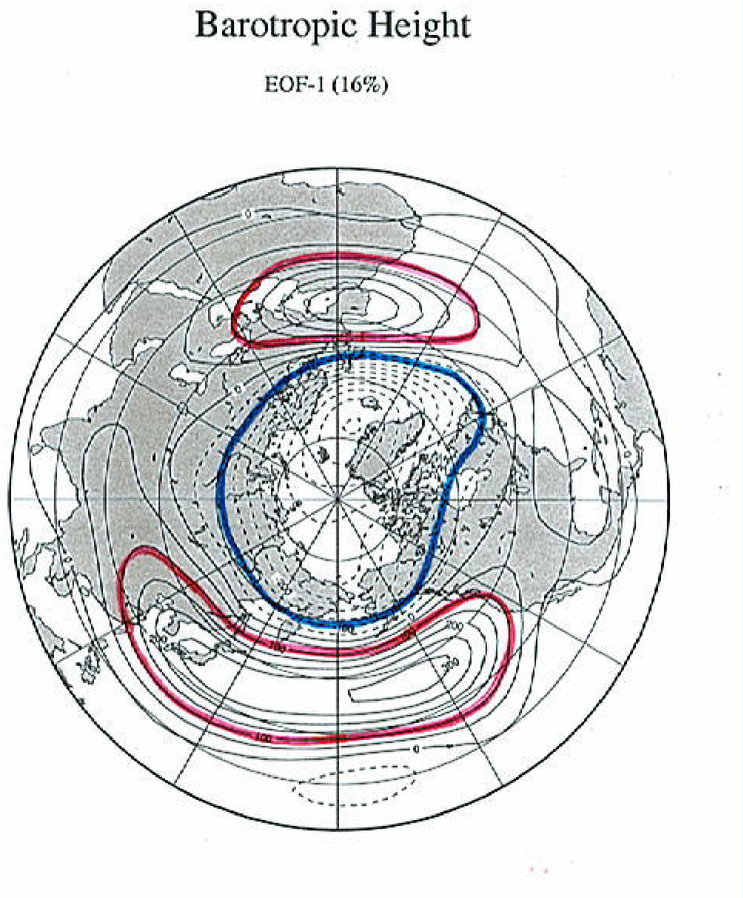
Arctic Oscillation

Barotropic height (EOF-1)

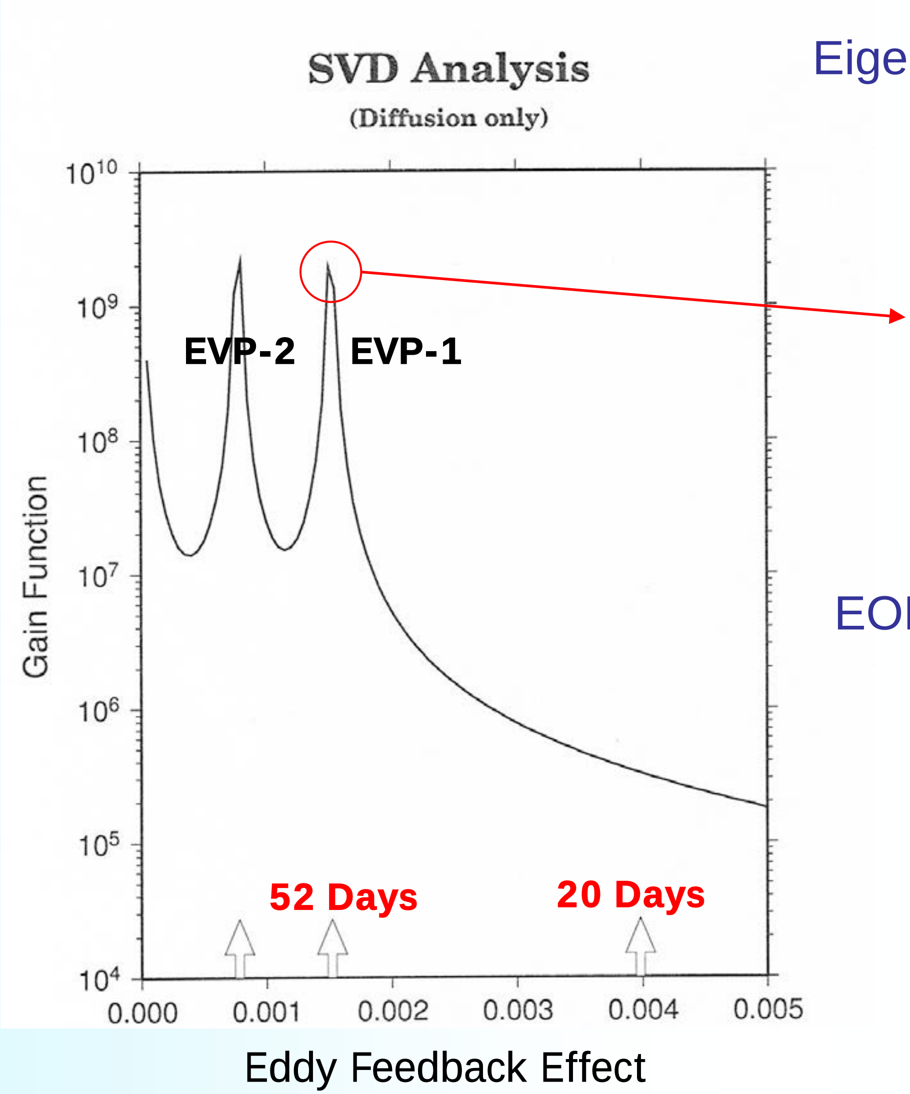
NCEP/NCAR

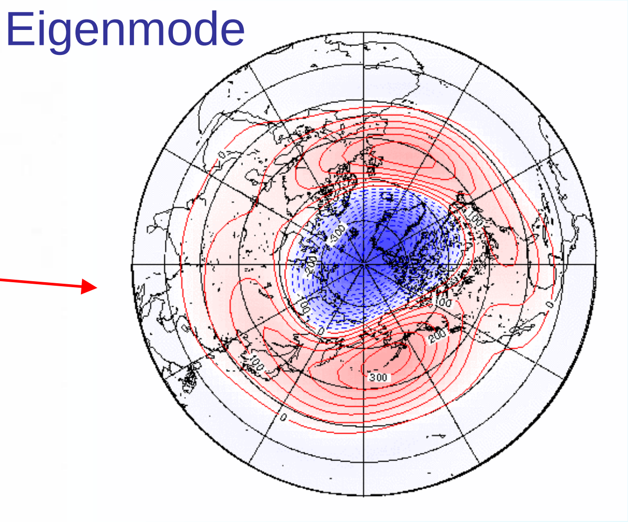


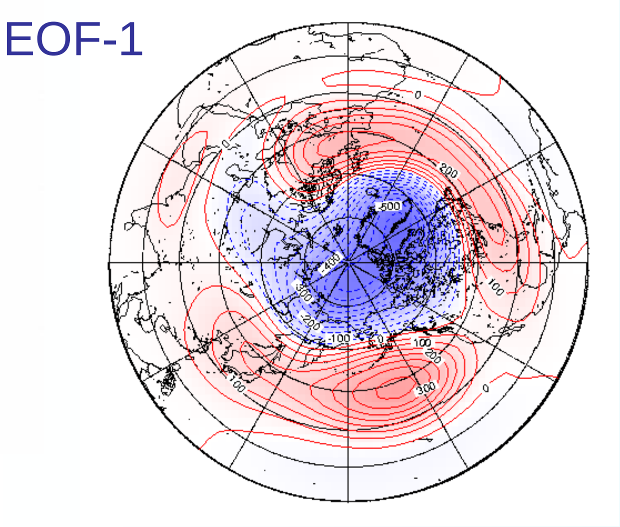
Barotropic S-Model



Singular Eigenmode Theory of AO

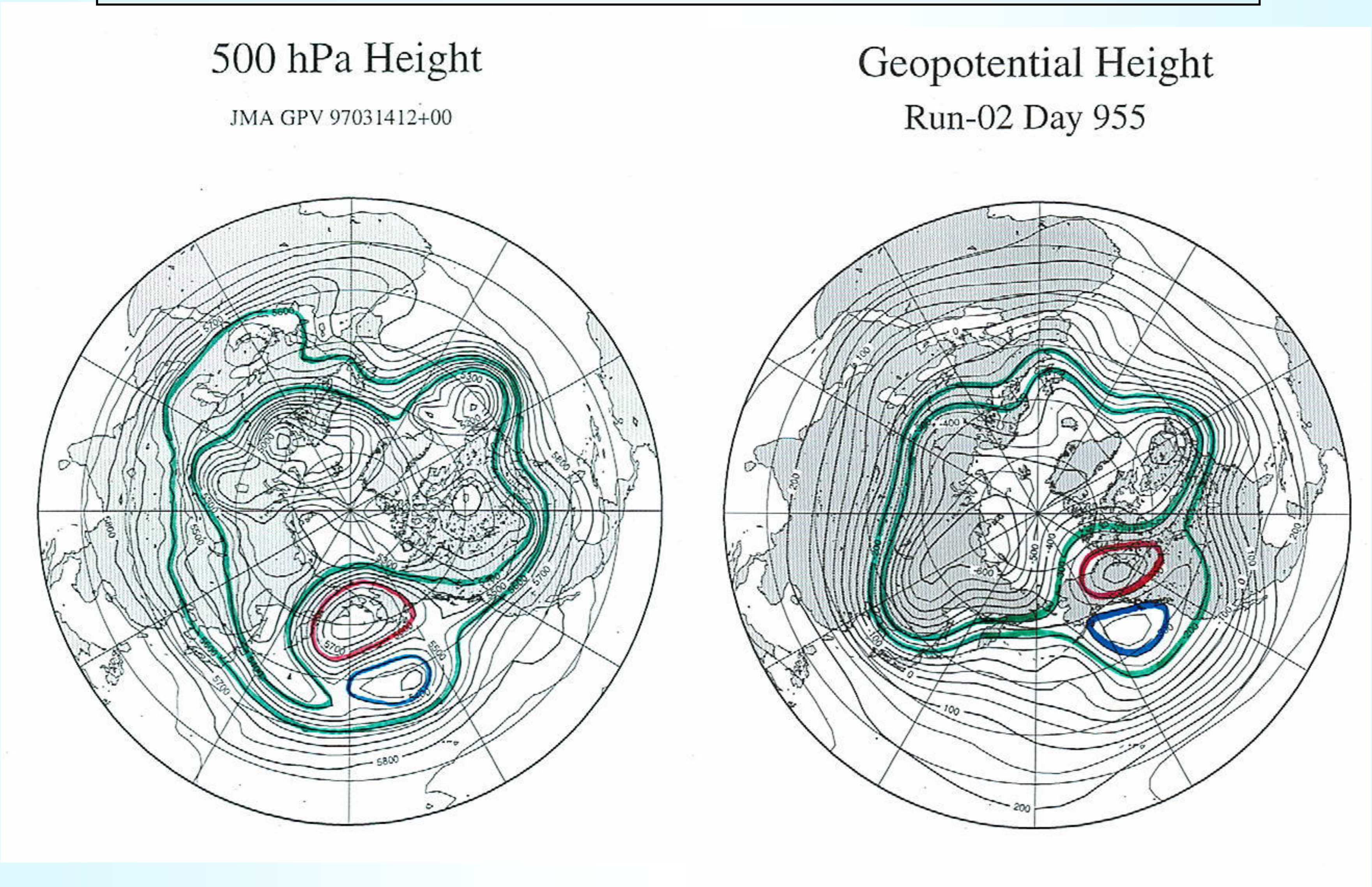




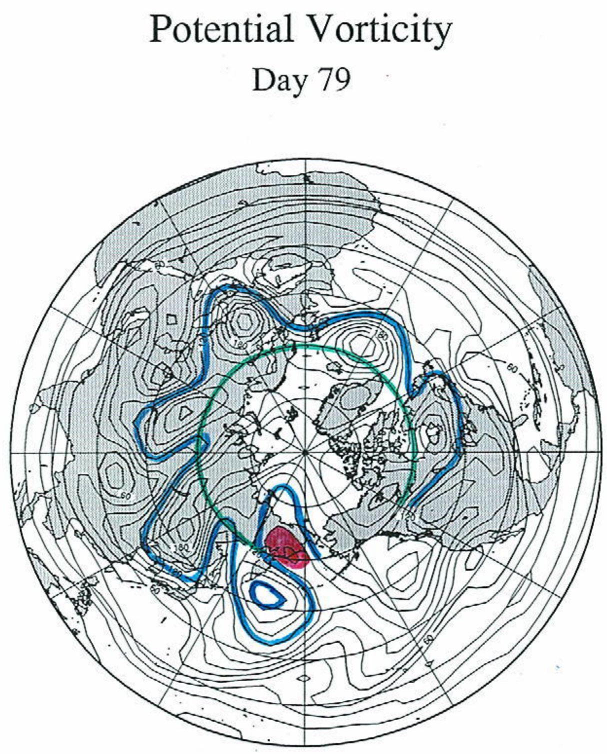


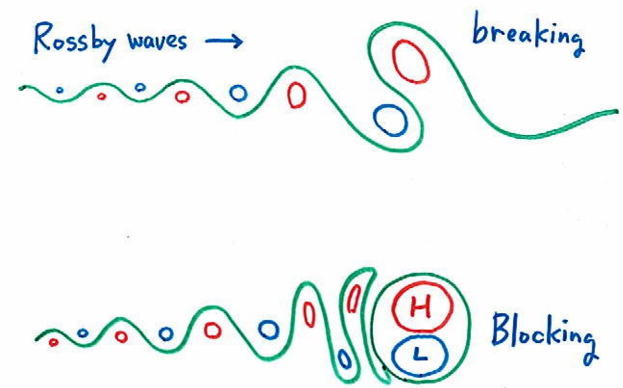
Tanaka and Matsueda (2005)

Blocking in the model

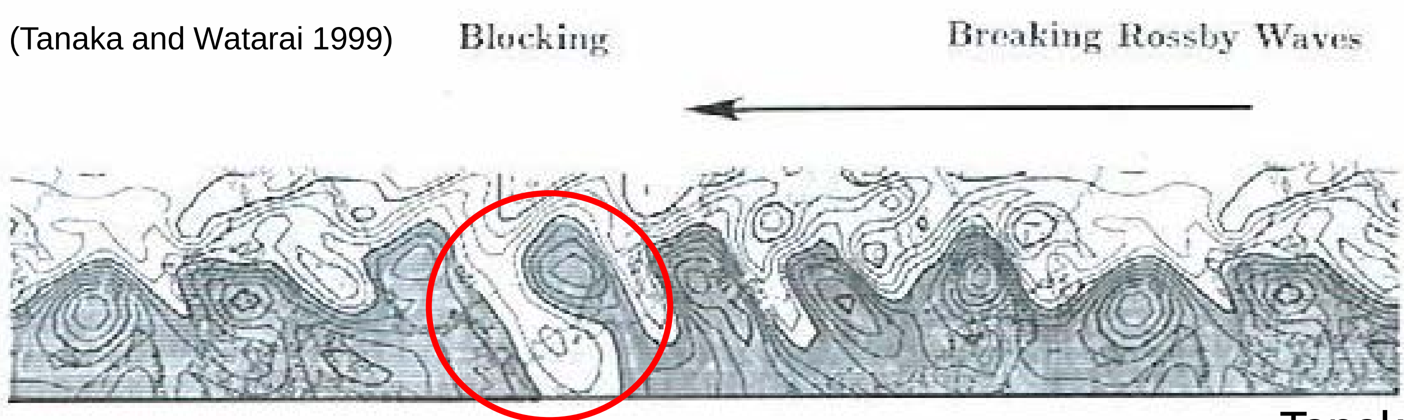


Blocking formation by Rossby wave breaking



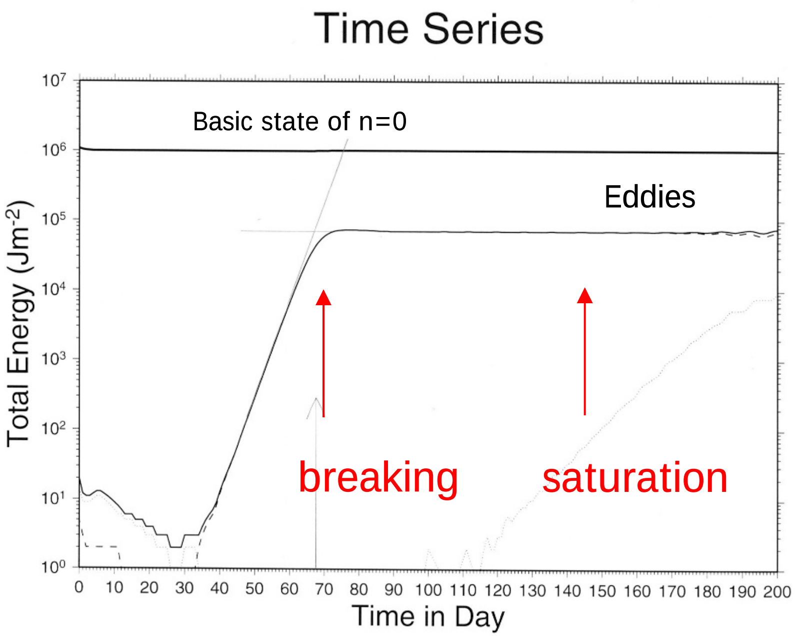


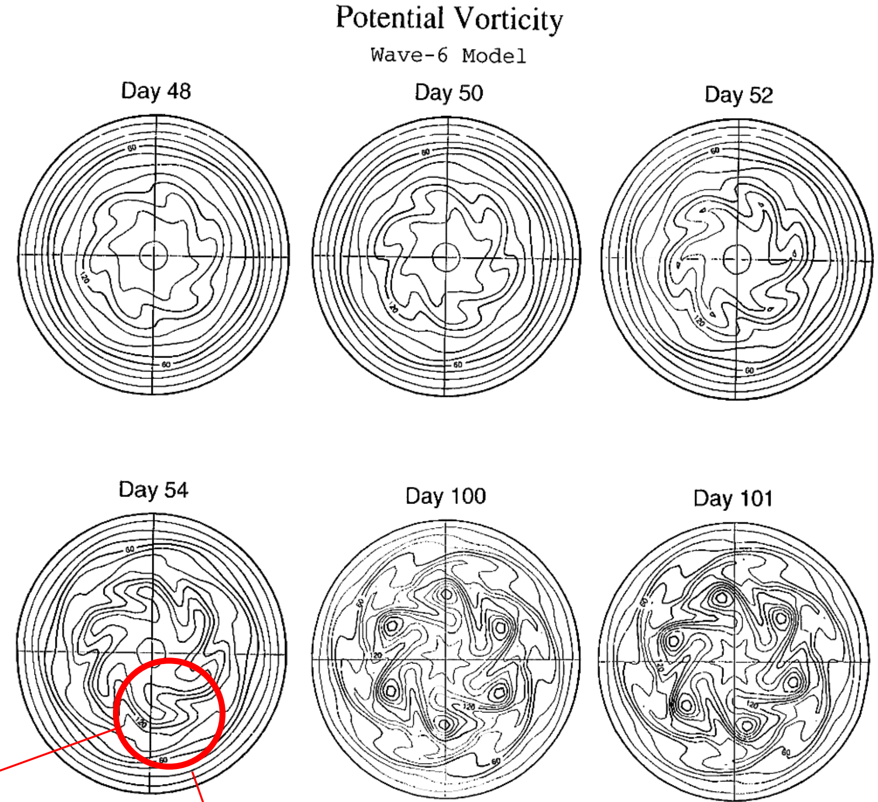
(Tanaka and Watarai 1999)

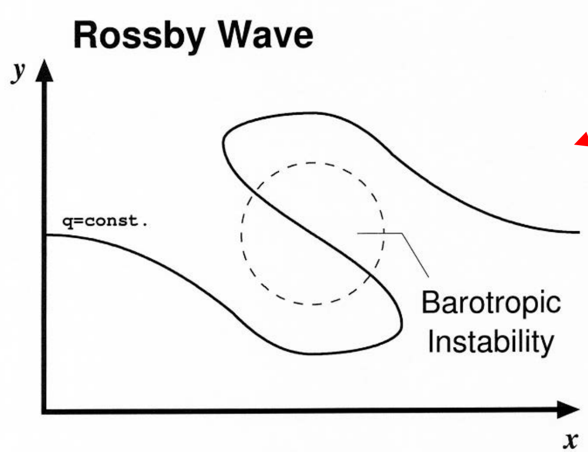


Tanaka (1998)

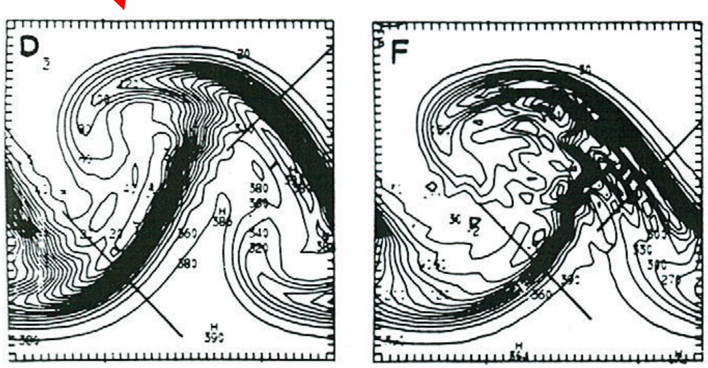
Rossby wave breaking and saturation







Tanaka and Watarai (1999)

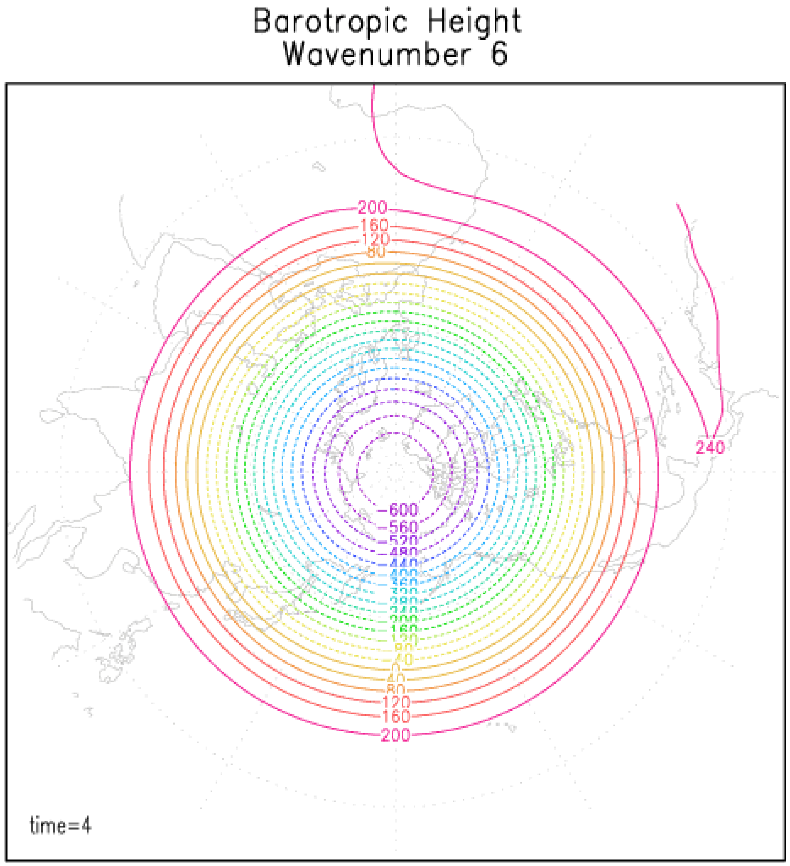


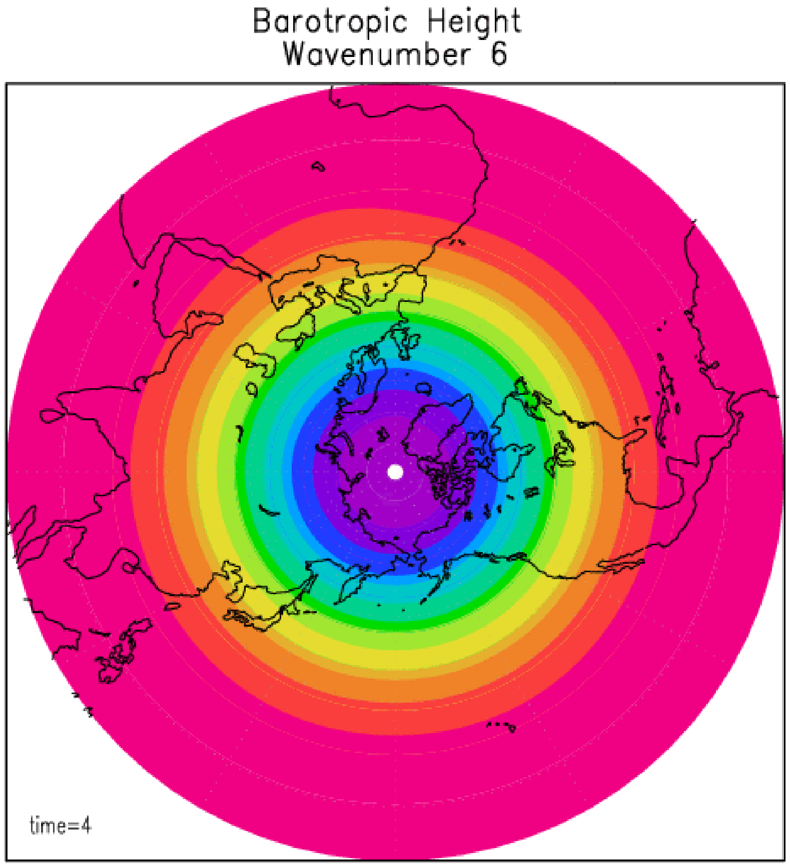
Mudrick (1974)

Zonalization

Rossby wave breaking for n=6

Growthrate × 1.7





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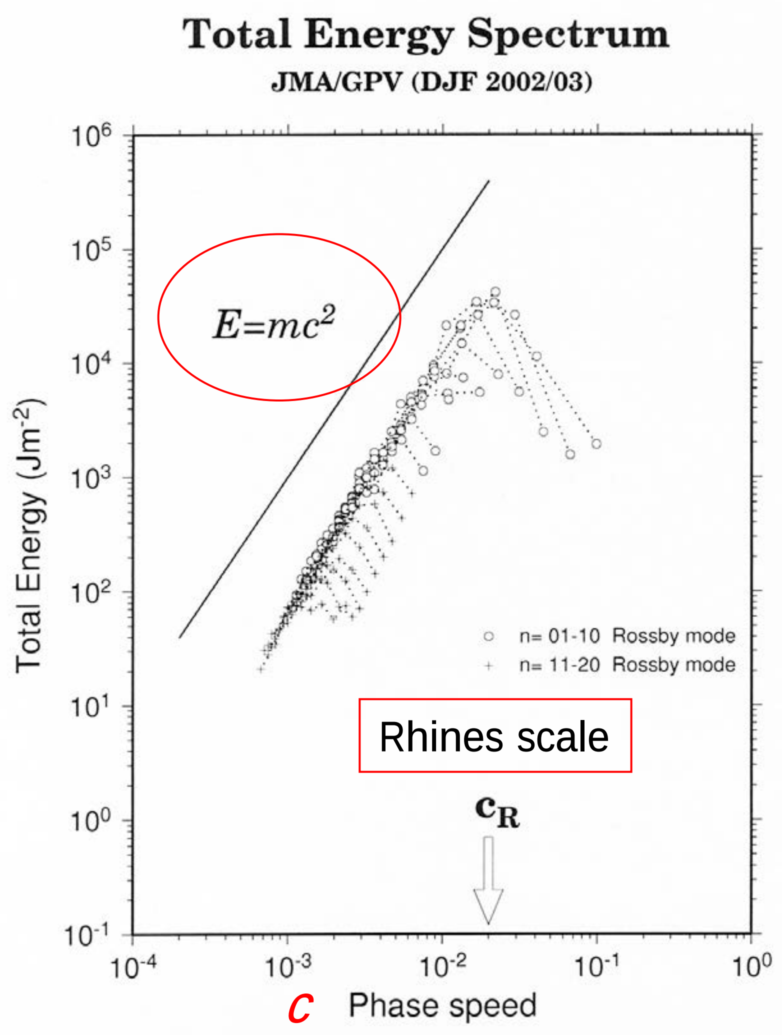
3D energy spectrum



By 3D normal mode expansion

$$\frac{dw_i}{d\tau} = -i\sigma_i w_i - i\sum_{jk} r_{ijk} w_j w_k + f_i$$

$$E_i = \frac{1}{2} p_s h_0 \,|\, w_i \,|^2$$

$$c_i = \sigma_i / n$$ Phase speed

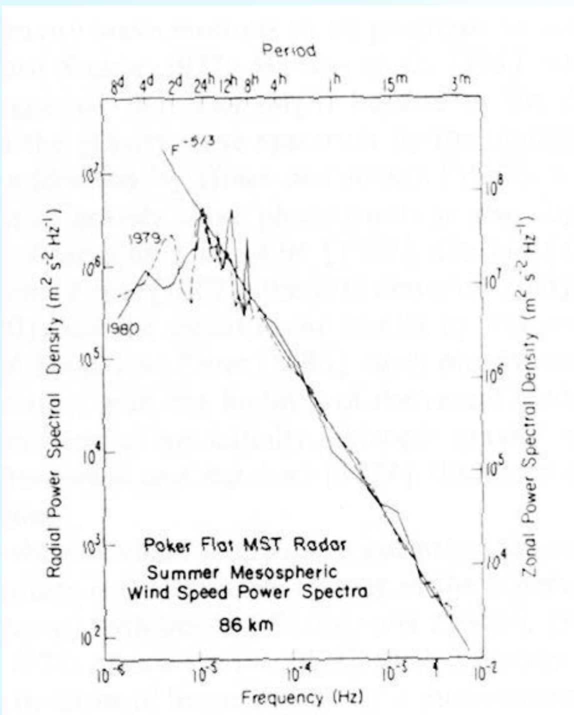
$$E = mc^2$$?

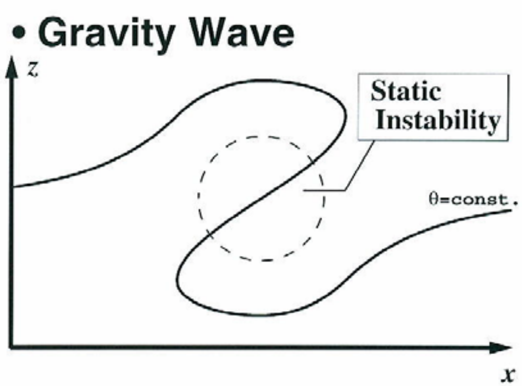
Tanaka et al. (2004 GRL)

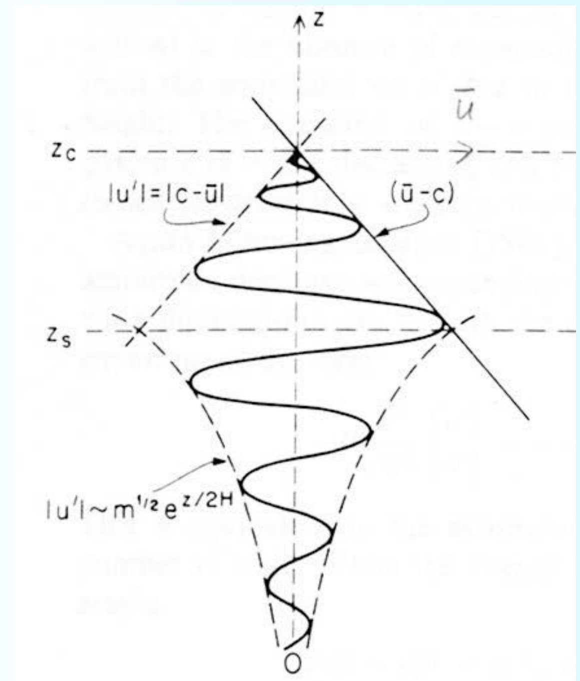
Saturation theory in gravity waves

Saturation spectrum

Breaking gravity waves







-3/5 power

$$\frac{\partial \theta}{\partial z} < 0$$

Breaking condition

Fritts (1984)

Saturation theory in Rossby waves

$$\frac{\partial q}{\partial y} < 0, \qquad q = \nabla^2 \psi + f$$

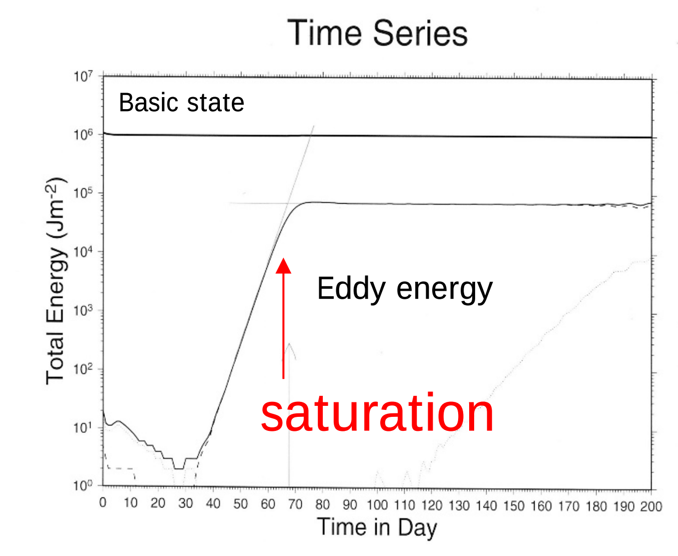
Tanaka and Watarai (1999)

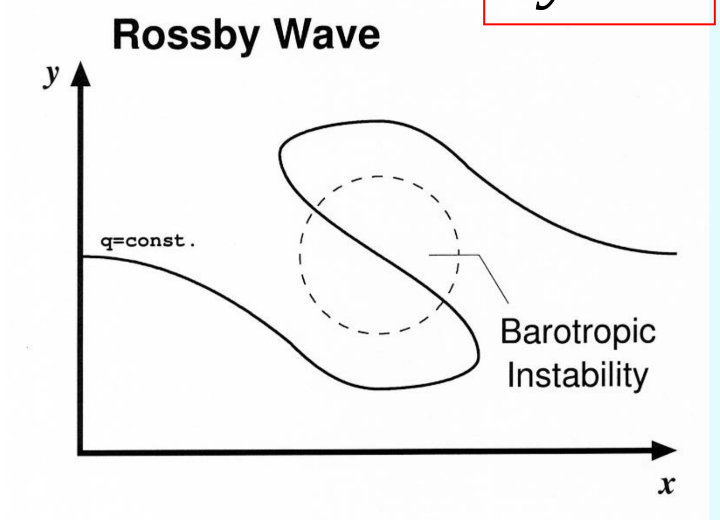
PV in barotropic model

$$\frac{\partial}{\partial y}(\nabla^2 \psi + f) = -\nabla^2 u + \beta \quad < 0$$

$$u < -\frac{\beta}{n^2 + l^2} = c$$

Breaking condition $\frac{\partial q}{\partial y} < 0$





Saturation energy spectrum

$$u < -\frac{\beta}{n^2 + l^2} = c$$

$$|u| \approx |v|$$

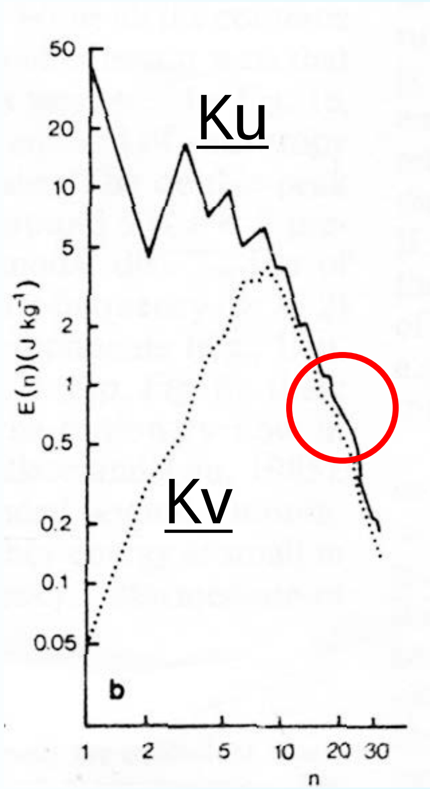
(Tanaka and Kasahara 1992)

$$E = \frac{1}{g}\int_0^{p_s} \frac{1}{2}(u^2 + v^2)dp$$

$$= \frac{p_s}{g}c^2 = mc^2$$

$$m = p_s / g$$

Mass for unit area

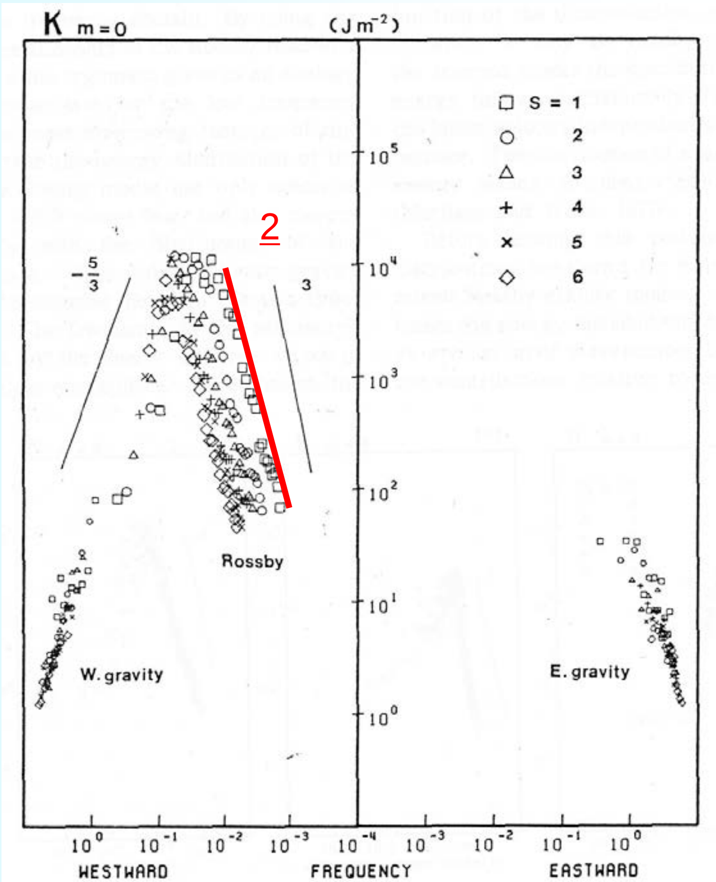


$$\frac{\partial q}{\partial y} < 0 \quad \Rightarrow \quad E = mc^2$$

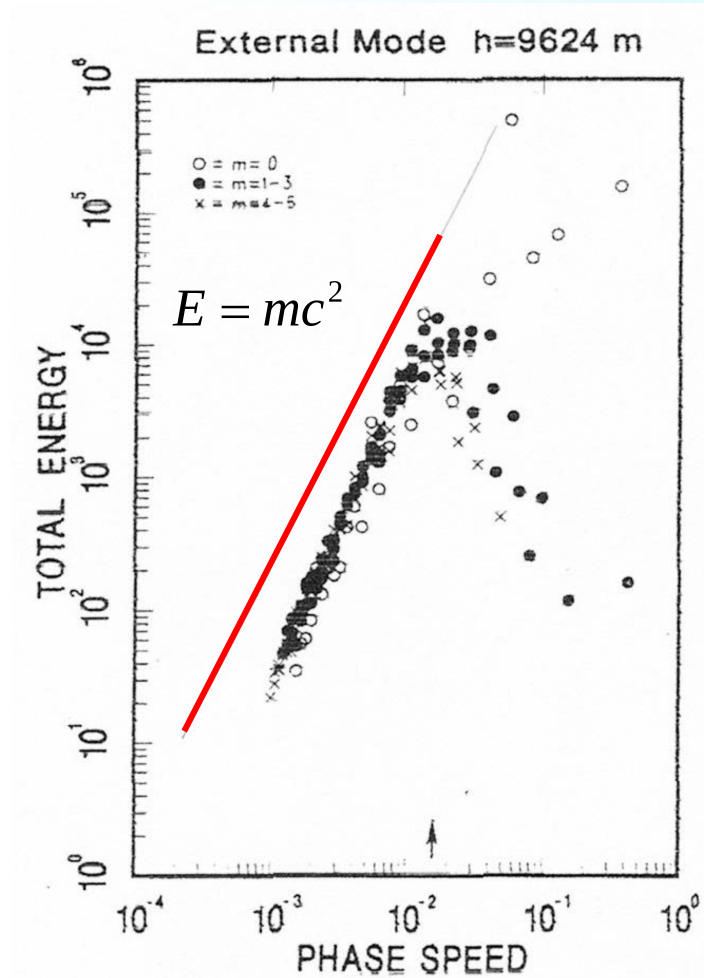
Shepherd (1987)

Observed energy spectrum in *c*-domain

FGGE SOP1



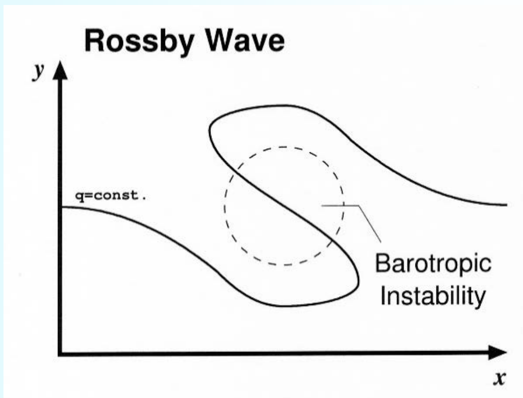
Tanaka (1985)



Tanaka and Kung (1988)

Global energy spectrum of $E = mc^2$

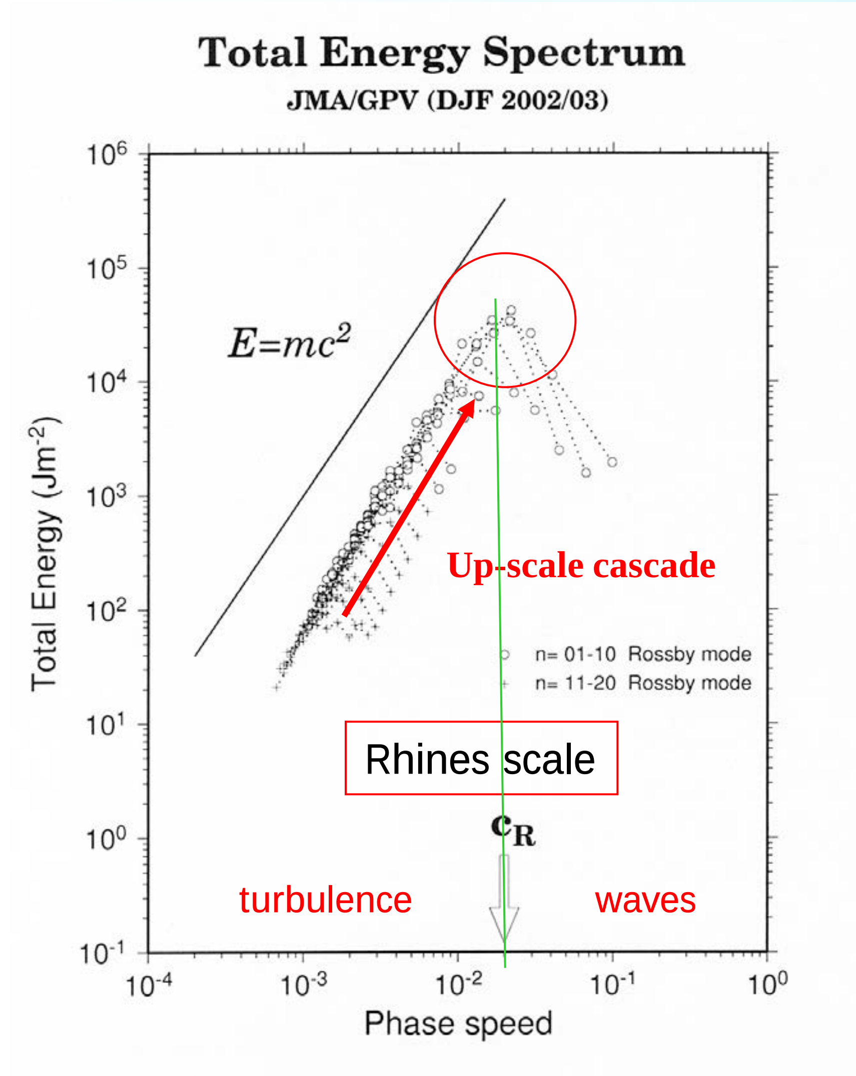
(Tanaka et al. 2004 GRL)



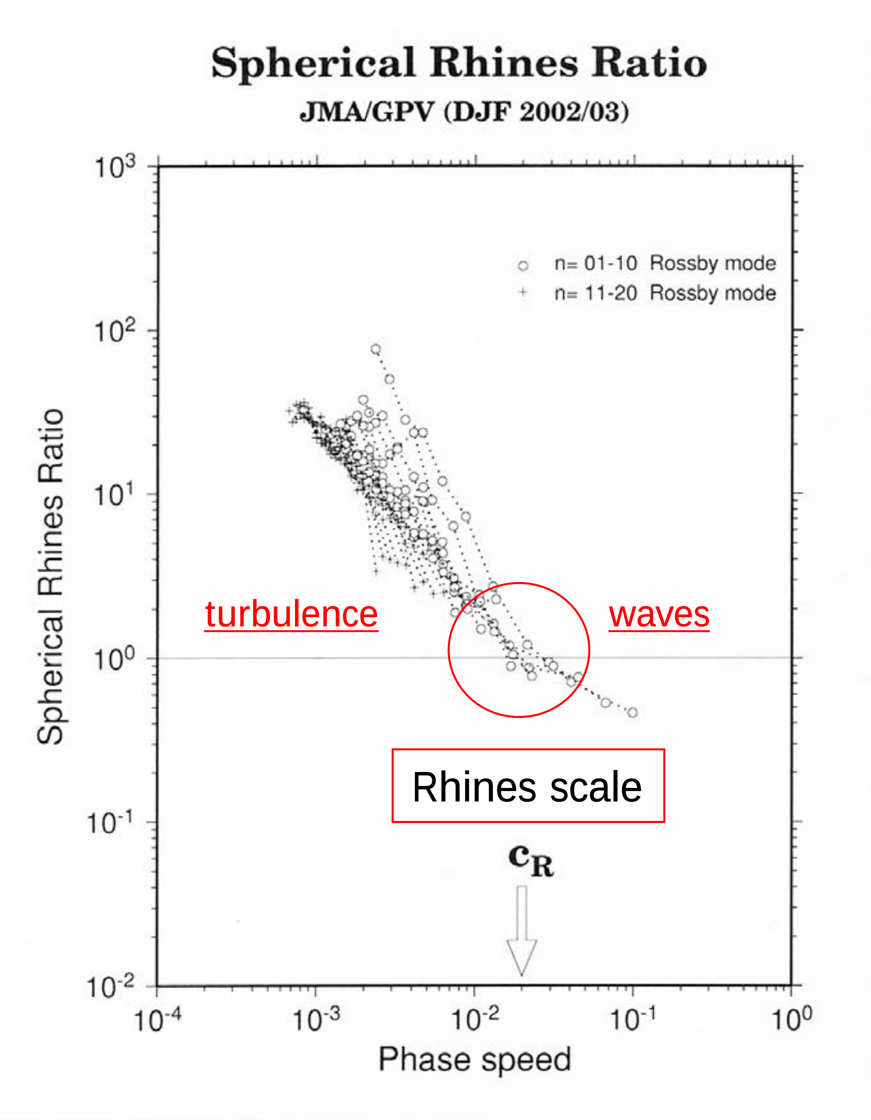
$$\frac{\partial q}{\partial y} < 0 \quad \Rightarrow \quad E = mc^2$$

c Rossby phase speed

$m = p_s / g$ Mass of the air



Rhines sclae on a sphere



$$\frac{dw_i}{d\tau} = -i\sigma_i w_i - i\sum_{jk} r_{ijk} w_j w_k + f_i$$

$$R_i = \frac{|\sum r_{ijk} w_j w_k|}{|\sigma_i w_i|}$$ Rhines ratio

turbulence $R_i > 1$

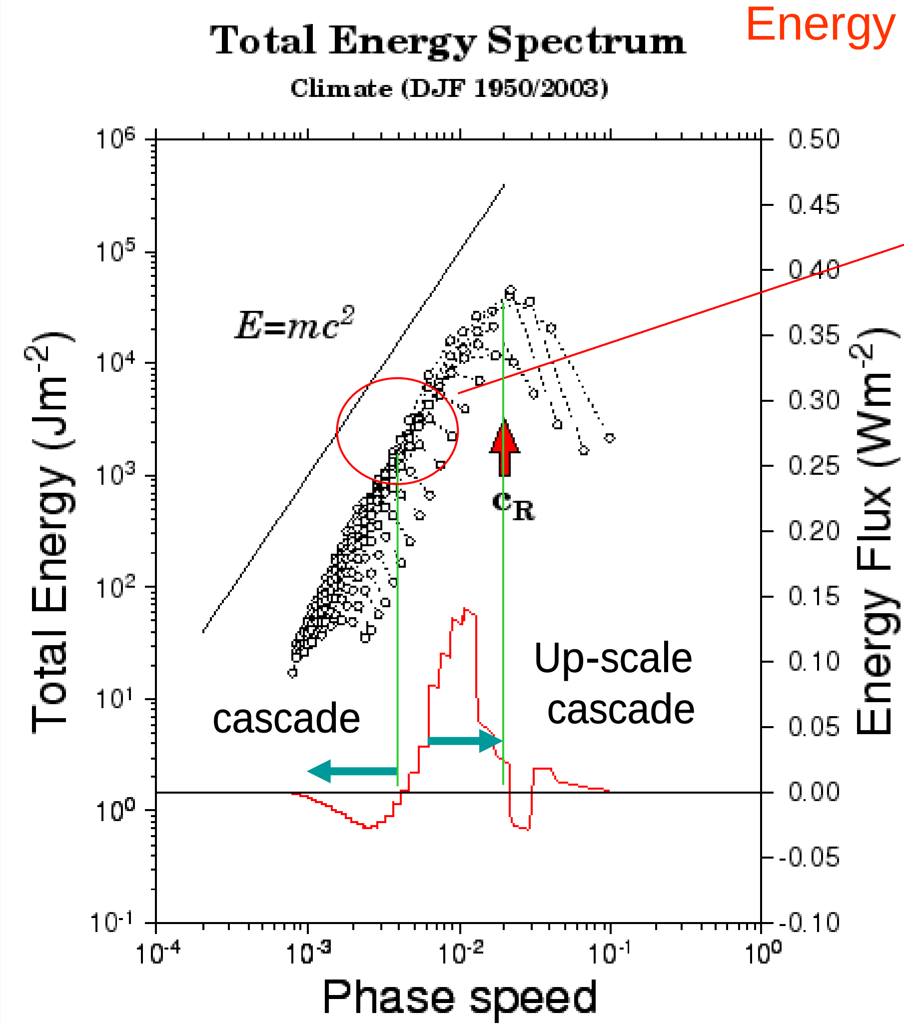
waves $R_i < 1$

Rhines scale $R_i = 1$

（Tanaka et al. 2004 GRL)

Energy flux in *c*-domain

Energy accumulates at phase speed 0



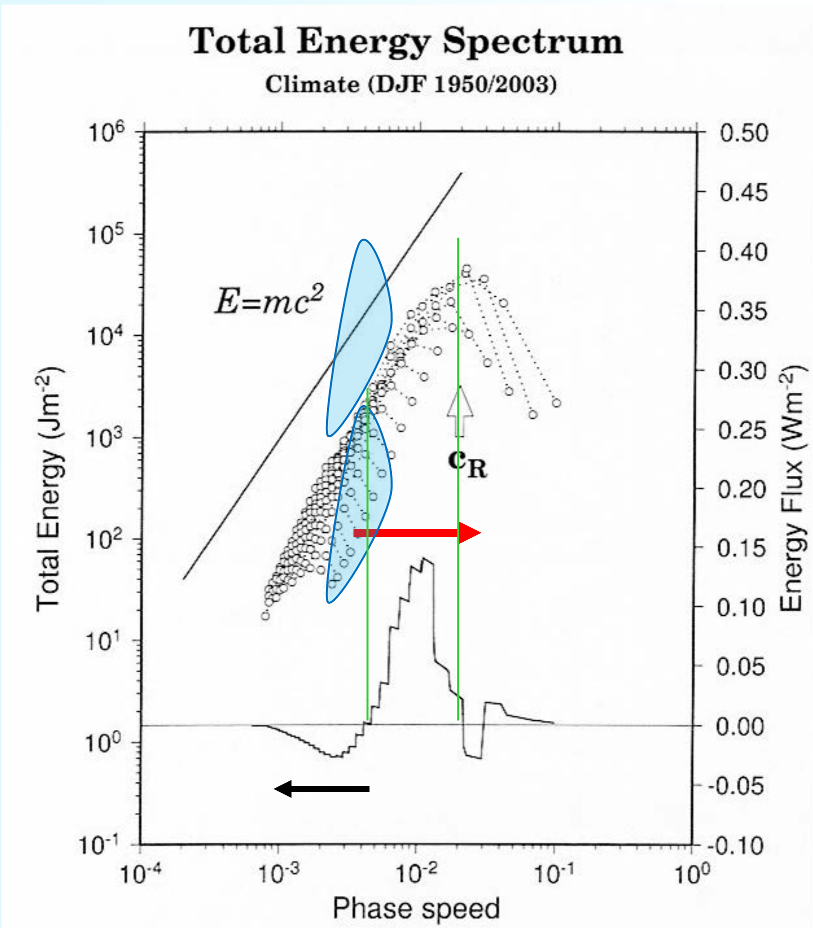
wave-wave

Source of energy

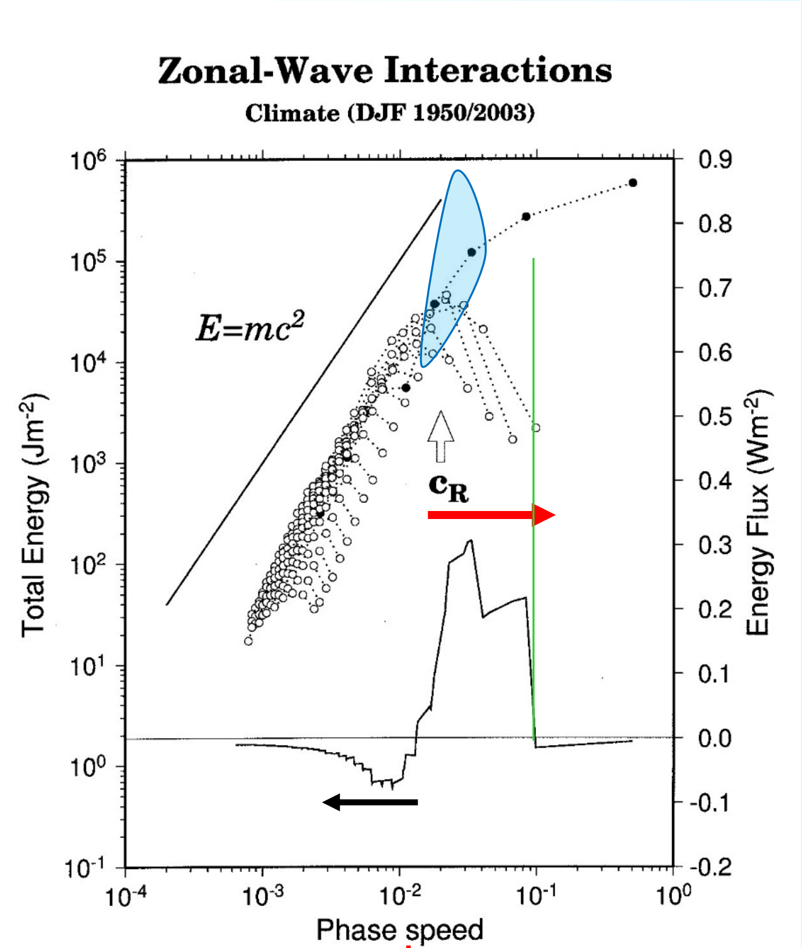
There is an energy source in the middle of the spectrum: inertial subrange theory fails.

Theory of Rossby wave saturation

Excitation of blocking and AO by up-scale energy cascade

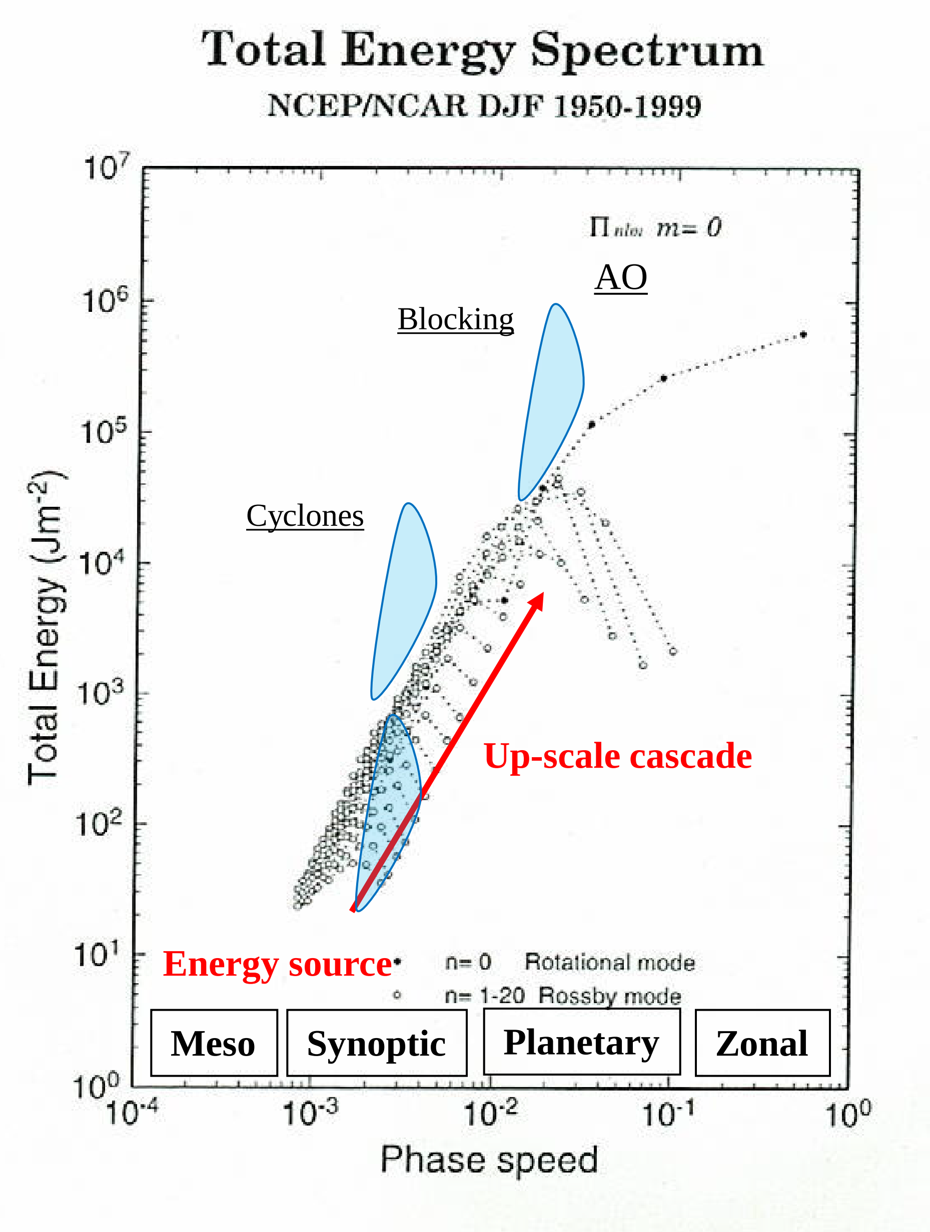


wave-wave

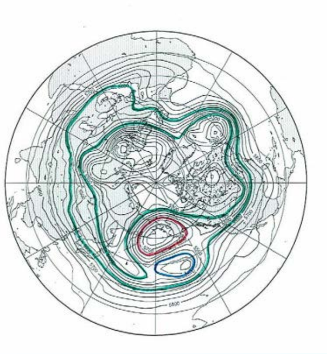
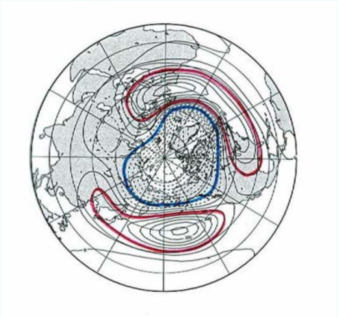
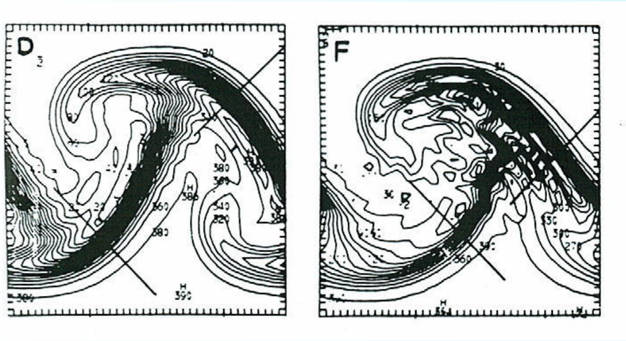


zonal-wave

(Tanaka and Terasaki 2004)



Low-frequency variability of the atmosphere

Summary

(1) Energy spectrum is examined in the 3D wavenumber domain using phase speed *c*.

(2) Energy spectrum of $E=mc^2$ is obtained and explained by Rossby wave saturation

(3) Up-scale energy cascade to Rhine's scale forms blocking

(4) Further up-scale cascade to zonal energy forms the Arctic Oscillation

END

Thank you.

Barotropic S-Model

○ 3-D Spectral Model

$$\frac{dw_i}{d\tau} + i\sigma_i w_i = -i \sum_{jk} r_{ijk} w_j w_k + f_i, \qquad i = 1, 2, 3, \ldots, \tag{7}$$

where the symbols denote:
w_i : expansion coefficient of U
f_i : expansion coefficient of F
σ_i : Laplace's tidal frequency
r_{ijk} : nonlinear interaction coefficient
τ : dimensionless time

○ Barotropic Spectral Model

$$\frac{dw_i}{d\tau} + i\sigma_i w_i = -i \sum_{jk} r_{ijk} w_j w_k + s_i, \qquad i = 1, 2, 3, \ldots, \quad (m = 0), \tag{8}$$

where the external forcing s_i includes barotropic-baroclinic interactions.

Model description

Barotropic S-Model

Barotropic forcing

○ External Forcing

External forcing s_i is statistically obtained by observed data using the least square method:

$$s_i = \tilde{s}_i + \mathbf{A}_{ij} w_j + \mathbf{B}_{ij} w_j^* + \epsilon_i'. \tag{9}$$

where $\tilde{s}_i$ is the climate of s_i and the matrices $\mathbf{A}_{ij}$ $\mathbf{B}_{ij}$ are evaluated by minimizing ϵ_i:

$$\mathbf{A}_{ij} = \overline{s_i' w_j^+}. \tag{10}$$

where $s_i' = s_i - \tilde{s}_i$ and pseudo-inverse of w_j is

$$w_j^+ = w_k^H \, \overline{(w_k w_j^H)}^{-1} \tag{11}$$

The matrix $\mathbf{B}_{ij}$ is similarly obtained stepwise by minimizing the first residual δ_i

$$\mathbf{B}_{ij} = \overline{\delta_i w_j^{*+}}, \tag{12}$$

Finally, the external forcing is given by

$$s_i = \tilde{s}_i + \mathbf{A}_{ij} w_j + \mathbf{B}_{ij} w_j^* + (BC)_{ij} w_j \tag{13}$$

$$+ (DF)_{ij} w_j + (DZ)_{ij} w_j + (DE)_{ij} w_j. \tag{14}$$

Physical processes considered in the S-Model:
(BC): baroclinic instability
(DF): biharmonic diffusion
(DZ): zonal surface stress and
(DE): Ekman pumping.